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Strong regularity of affine cocycles over irrational rotations

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Contents

0.1 Introduction

0.1.1 General notions: automorphism, Koopman operator, isomorphism, factor

By a *dynamical system* we mean a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability standard Borel space and T is a bimeasurable ($T^{-1}\mathcal{B} = T\mathcal{B} = \mathcal{B}$) bijection of X which is measure-preserving ($\mu(A) = \mu(T^{-1}A) = \mu(TA)$ for each $A \in \mathcal{B}$). Then T is called an *automorphism* of $(X, \mathcal{B}, \mu)$ and we will often write $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$. By $Aut(X, \mathcal{B}, \mu)$ we denote the group of all automorphisms of $(X, \mathcal{B}, \mu)$. Each $T \in Aut(X, \mathcal{B}, \mu)$ determines a unitary operator U_T (called the *Koopman operator* associated to T) of $L^2(X, \mathcal{B}, \mu)$: $U_T(f) = f \circ T$ for each $f \in L^2(X, \mathcal{B}, \mu)$.

In order to classify dynamical systems, we usually use metric isomorphism (the measure-theoretic isomorphism). Recall that $T_i \in Aut(X_i, \mathcal{B}_i, \mu_i), i = 1, 2$, are said to be *metrically isomorphic* if there exists an isomorphism $S : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ (of probability spaces) such that $S \circ T_1 = T_2 \circ S$. If S is only assumed to be (a.e.) surjective, measurable and "measure-preserving" then S is called a *homomorphism* and T_2 is called a *factor* of T_1.

0.1.2 Equivalence notions and relations between them: metric, weak and spectral isomorphism, Markov quasi-similarity

It follows that measure-theoretic isomorphism implies *spectral equivalence (isomorphism)* of the corresponding unitary operators; indeed, $U_{S^{-1}} : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$, $U_{S^{-1}}f = f \circ S^{-1}$ for $f \in L^2(X_1, \mathcal{B}_1, \mu_1)$ settles a unitary equivalence of U_{T_1} and U_{T_2}. It is a classical fact that the converse does not hold, see [4] for historically one of the first relevant examples[1]. Other classical examples of spectrally isomorphic and metrically non-isomorphic dynamical systems arise when we consider the class of Bernoulli shifts: all of them are spectrally isomorphic while the entropy classify them measure-theoretically (see [38]).

We now present two other concepts of equivalence of dynamical systems situated between spectral isomorphism and measure-theoretic isomorph-

[1] These are examples as in (0.1.2) below. If we take the automorphism $\overline{T}$ defined in (0.1.2) and consider $\widetilde{T} : (x, y) \mapsto (x+\alpha, 2x+y)$ then $\overline{T}$ and $\widetilde{T}$ are spectrally isomorphic but are not metrically isomorphic. It is a particular case of spectral isomorphism in the class of quasi-discrete spectrum automorphisms considered in Chapter 3: two automorphisms with quasi-discrete spectrum are spectrally isomorphic if and only if they have the same discrete spectrum.

ism. In [47], Sinai introduced the notion of weak isomorphism. Two automorphisms $T_i \in Aut(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, are said to be *weakly isomorphic* if there exist homomorphisms $S_1 : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ and $S_2 : (X_2, \mathcal{B}_2, \mu_2) \to (X_1, \mathcal{B}_1, \mu_1)$ such that $S_1 \circ T_1 = T_2 \circ S_1$, $S_2 \circ T_2 = T_1 \circ S_2$, i.e. T_1 and T_2 are factors of each other. Clearly, metric isomorphism implies weak isomorphism while the converse does not hold; for relevant examples, see e.g. [23], [42], [43], [49]. It is already shown in [47] that weak isomorphism implies spectral isomorphism[2].

In order to introduce the fourth concept of equivalence of automorphisms, first, recall that a linear contraction

$$\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$$

is called a *Markov operator* if $\Phi\mathbb{1} = \mathbb{1} = \Phi^*\mathbb{1}$ and Φ "preserves" the cone of non-negative functions ($\Phi f \geqslant 0$ whenever $0 \leqslant f \in L^2(X_1, \mathcal{B}_1, \mu_1)$).

In [50], Vershik introduced a new concept for classification of dynamical systems by considering the notion of Markov quasi-similarity: two automorphisms $T_i \in Aut(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, are called *Markov quasi-similar* (MQS) if there are Markov operators

$$\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2), \quad \Psi : L^2(X_2, \mathcal{B}_2, \mu_2) \to L^2(X_1, \mathcal{B}_1, \mu_1)$$

both with dense ranges, satisfying the intertwining conditions[3]

$$\Phi \circ U_{T_1} = U_{T_2} \circ \Phi, \quad \Psi \circ U_{T_2} = U_{T_1} \circ \Psi.$$

Clearly, each Koopman operator is also Markov but not vice-versa. If however, a Markov operator Φ intertwinning T_1 and T_2 is unitary, then it is Koopman, i.e. $\Phi = U_{S^{-1}}$, where S settles an isomorphism of T_1 and T_2 (see e.g. [29], [50]). It is easy to see that weak isomorphism implies Markov quasi-similarity (indeed, $U_{S_1}^{-1}$, $U_{S_2}^{-1}$ yield a Markov quasi-similarity). Moreover, Markov quasi-similarity implies spectral isomorphism, see e.g. [13][4].

The relations between the four notions of equivalence can be now summarized as follows:

[2]The converse again is false: the automorphisms $\overline{T}$ and $\widetilde{T}$ mentioned in footnote 1 are spectrally isomorphic but are not weakly isomorphic. In fact, weak isomorphism implies isomorphism in the class of quasi-discrete spectrum automorphisms (see Remark 1.6.1 and Theorem 1.9.4).

[3]Similarly to the notion of factor, if only Φ above exists then one says that T_2 is a *Markov quasi-factor* (MQF) of T_1. Recall that an MQF of an ergodic T is ergodic ([13]).

[4]It is also noted in [13] that $T_1 = T \times B$, where T is an irrational rotation, B a Bernoulli automorphism and $\overline{T}$ defined in (0.1.2) are spectrally isomorphic but are not Markov quasi-similar.

$$\begin{array}{ccccccc} \text{metric} & \xrightarrow{(r1)} & \text{weak} & \xrightarrow{(r2)} & \text{MQS} & \xrightarrow{(r3)} & \text{spectral} \\ \text{isomorphism} & & \text{isomorphism} & & & & \text{isomorphism} \end{array}$$

0.1.3 Vershik's question on Markov quasi-similarity. Markov quasi-factors of quasi-discrete spectrum automorphisms

We have already noticed that $(r1)$ and $(r3)$ cannot be reversed and it was one of main questions by Vershik in [50] whether $(r2)$ can be reversed. The negative answer to this question was given recently in [13]. The construction in [13] is given in the class of so called compact Abelian group extensions of (ergodic) rotations. In connection with it two natural questions arise.

First of all, the examples from [13] are not weakly mixing, so one can ask whether the negative answer to Vershik's question can be obtained in a class of transformations with better mixing properties. The second group of questions arises when we think about finding simpler (or more "natural") examples of MQS automorphisms which are not weakly isomorphic. Such examples, of course, cannot be found in the class of discrete spectrum automorphisms (ergodic rotations) as here already spectral isomorphism and metric isomorphism coincide[5] but the problem seems to be completely open in the class of automorphisms with quasi-discrete spectrum[6] (the latter class of automorphisms was introduced by Abramov in [2]). To handle such a problem, it seems to be natural, first, to understand Markov quasi-factors of an automorphism.

We will not really deal with the latter problem in the thesis, although we will notice[7] in Chapter 3 (see Theorem 3.1.4) that Markov quasi-factors of quasi-discrete spectrum automorphism have quasi-discrete spectrum. The theorem is a generalization of a classical result by Hahn and Parry [17] saying that every factor of an automorphism with quasi-discrete spectrum also has quasi-discrete spectrum. The problem of Markov quasi-factors will be rather a motivation for us to see some further relations with the theory of joinings and the theory of ergodicity of affine cocycles.

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[5] This is the classical Halmos-von Neumann Theorem, see e.g. [11].

[6] For the definition, see Section 1.9.2.

[7] The result will be a consequence of the proof of the theorem by E. Lesigne [33] characterizing quasi-discrete spectrum of a given order.

0.1.4 Joinings and Markov quasi-factors

In order to see a relationship of MQS with the theory of joinings, notice that each Markov operator

$$\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2), \quad \Phi \circ U_{T_1} = U_{T_2} \circ \Phi$$

via the formula

$$\int_{B_2} \Phi(\mathbb{1}_{B_1}) \, d\mu_2 = \int_{X_1 \times X_2} \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} \, d\rho \qquad (0.1.1)$$

determines a probability measure ρ on $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ such that:

(i) its marginals on X_1, X_2 are μ_1 and μ_2 respectively,

(ii) ρ is $T_1 \times T_2$-invariant

(see e.g. [15]).

Each measure satisfying (i) and (ii) is called a *joining* of T_1 and T_2. In fact, there is a natural correspondence between Markov intertwining operators and joinings (given by (0.1.1)). Hence, MQS requires the existence of special joinings between two dynamical systems. To see deeper relations, recall that given $T_i \in Aut(X_i, \mathcal{B}_i, \mu_i)$, $i \geqslant 1$, by a joining of all these automorphisms one means a probability measure ρ on $(X_1 \times X_2 \times \ldots, \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \ldots)$ which is $T_1 \times T_2 \times \ldots$-invariant and has all one-dimensional marginals "correct" (as in (i)). Note that each joining ρ yields a new automorphism $T_1 \times T_2 \times \ldots \in Aut(X_1 \times X_2 \times \ldots, \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \ldots, \rho)$. When all of T_i are equal to T, one speaks about infinite self-joinings of T.

Here is the main result relating joinings and the theory of MQF ([13], see also footnote 3):

Theorem 0.1.1. *Let T be an ergodic automorphism. If S is a Markov quasi-factor of T then S is a (classical) factor of some infinite ergodic self-joining of T.*

0.1.5 Basic affine automorphism of $\mathbb{T}^2$

Coming back to quasi-discrete spectrum automorphisms, recall that the simplest example from this class (which is not with discrete spectrum) is the transformation $\overline{T}$ of the additive torus $\mathbb{T}^2 = [0, 1)^2$ (considered with Lebesgue measure) given by the formula

$$\overline{T}(x, y) = (x + \alpha, x + y), \qquad (0.1.2)$$

where α is irrational. Even though the form of ergodic joinings (of all orders) for $\overline{T}$ is known [8] and the form of factors of infinite ergodic self-joinings of $\overline{T}$ is also known, the problem whether MQF of $\overline{T}$ are its classical factors is open.

If we want to describe ergodic self-joinings of $\overline{T}$, we study the ergodic decomposition of the transformations of $\mathbb{T}^{d+2}$ of the following form

$$(x, y_1, y_2, \ldots, y_{d+1}) \mapsto (x+\alpha, y_1+\phi'(x), y_2+\phi'(x+\beta_1), \ldots, y_{d+1}+\phi'(x+\beta_d))$$

with $\phi'(x) = x$. That is, we must study the ergodicity problem of the cocycle[9]

$$\Theta'_{d+1} : \mathbb{T} \to \mathbb{T}^{d+1}, \quad \Theta'_{d+1} = (\phi', \phi' \circ S_1, \ldots, \phi' \circ S_d),$$

where $S_i(x) = x + \beta_i$ $(i = 1, \ldots, d)$, and its ergodicity depends whether some cohomological equations admit measurable solutions (see Chapter 2 for details).

0.1.6 Real-valued cocycles, Rokhlin cocycles - toward the main problems of dissertation

The situation becomes more complicated when we look at ϕ' as a real-valued cocycle. In order to do it, we replace ϕ' by $\phi(x) = x - \frac{1}{2}$ $(\phi : \mathbb{T} \to \mathbb{R})$[10]. We now ask whether

$$\Theta_{d+1} : \mathbb{T} \to \mathbb{R}^{d+1}, \quad \Theta_{d+1} = (\phi, \phi \circ S_1, \ldots, \phi \circ S_d) \tag{0.1.3}$$

is ergodic or, more generally, regular.[11]

It is a classical fact that $\Theta_1 = \phi$ is ergodic for every irrational rotation (see e.g. [40]). In the paper [28], it was shown that Θ_2 is regular whenever α has bounded partial quotients.

The problem of regularity of cocycles (taking values in locally compact second countable (l.c.s.c.) but not compact Abelian groups G, for example $G = \mathbb{R}^{d+1}$) of the above form is still important in the theory of joinings. Indeed, assume that $T \in Aut(X, \mathcal{B}, \mu)$ is ergodic and let G be an Abelian l.c.s.c. group. Assume that $\varphi : X \to G$ is a cocycle and let $\mathcal{G} = (R_g)_{g \in G}$ be a (measurable[12], measure-preserving) G-action on a probability standard

[8]This is a particular case of so called compact Abelian group extensions of rotations for which joinings were described in [27].

[9] For the definition, see Subsection 1.8.

[10]In order to study ergodicity or regularity of integrable real-valued cocycle we must have them centered.

[11]For the definition, see 1.8.5.

[12]Measurability of a G-action means the continuity of all maps $G \ni g \mapsto \langle U_{R_g} f_1, f_2 \rangle \in \mathbb{C}$ for each $f_1, f_2 \in L^2(Y, \mathcal{C}, \nu)$.

Borel space $(Y, \mathcal{C}, \nu)$. Then the automorphism $T_{\varphi,\mathcal{G}} \in Aut(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ given by

$$T_{\varphi,\mathcal{G}}(x, y) = (Tx, R_{\varphi(x)}(y))$$

is called a *Rokhlin extension.*

It turns out that self-joinings of higher order of $T_{\varphi,\mathcal{G}}$ are strictly connected with cocycles of the form (0.1.3) above. Indeed, Theorem 3 in [28] gives a full description of all ergodic self-joinings of $T_{\varphi,\mathcal{G}}$ whenever cocycles $\Theta_{d+1} = (\phi, \phi \circ S_1, \ldots, \phi \circ S_d)$ are regular.

0.1.7 Strong regularity property

In what follows, we say that ϕ has the **strong regularity property** if all cocycles $\Theta = \Theta_{d+1} = \Theta_{d+1}(\alpha, S_1, \ldots, S_d)$ of the above form are regular (for all parameters $d, \alpha, S_1, \ldots, S_d$).

The problem of strong regularity of affine cocycles is the main object of study in the dissertation. One of the crucial observations (see Theorem 2.4.1) states that the diagonal subgroup $\Delta_{d+1} = \{(t, ..., t) : t \in \mathbb{R}\}$ is always included in the group of essential values of cocycle Θ. Due to this, the problem of strong regularity of affine cocycles is reduced to the problem of regularity of (vectorial) step cocycles of the form

$$\Phi_d(x) = (\mathbb{1}_{[0,\beta_j)} - \beta_j)_{j=1,\ldots,d}. \tag{0.1.4}$$

The problem of ergodicity or regularity of step functions, mainly in one dimensional case, has been broadly studied in the literature, for instance see [7], [8], [12], [30], [34], [37], [40]. We will generalize (see Theorem 0.1.2, $d = 2$) the recent result from [52], where it was proved that $\Phi = (\mathbb{1}_{[0,1/2)}(\cdot) - 1/2, \mathbb{1}_{[0,1/2)}(\cdot + \gamma) - 1/2)$ is regular for each $\gamma \in \mathbb{T}$ and α with bounded partial quotients.

0.1.8 Description of the content of the dissertation. Chapter 1

We now pass to a description of the content of the dissertation. In Chapter 1, we recall some basics in ergodic theory that will be needed in what follows: necessary facts from spectral theory, mixing and rigidity properties, induced automorphism, basic constructions (factors, extensions), coalescence and weak isomorphism, joinings and Markov operators, introduction to the theory of cocycles including the theory of Schmidt of essential values. We also recall a few examples of dynamical systems (irrational rotations, automorphisms with discrete and quasi-discrete spectrum, Gaussian systems) that will play special roles in the dissertation.

0.1.9 Description of the content of the dissertation. Chapter 2

Chapter 2, containing original results, has been written on the base of [10]. It begins by a further development of the theory of cocycles (Section 2.1) with some new results on essential values of cocycles with values in an Abelian l.c.s.c. group G and then in it is focused on the group $G = \mathbb{R}^d$. In Section 2.2, we present detailed combinatorial lemmas relating dynamical properties of an irrational rotation by α with the Diophantine approximation of α by rational numbers. In Section 2.3, we deal with the crucial problem of representation of (vectorial) step cocycles, in particular we introduce the notion of rational step cocycles. The main results can be summarized as follows.

Theorem 0.1.2. *Let Φ_d be as in (0.1.4).*

- *For $d = 1$, Φ_d is regular for every irrational rotation.*
- *For $d = 2$, if α has bounded partial quotients, then Φ_d is regular.*
- *For $d = 3$, if α has unbounded partial quotients, then there exists a choice of $\beta_1, \beta_2, \beta_3$ making Φ_d a non-regular cocycle.*

In Subsection 2.3.3, we also give sufficient conditions for the regularity of step cocycles using methods based on Diophantine properties of the values of the integrals for rational cocycles or Diophantine properties of the discontinuities of the cocycle.

Theorem 0.1.3. *Let Φ be a zero mean step function. If Φ has well separated discontinuities*[13]*, then the group of essential values $\mathcal{E}(\Phi)$ includes the set $\{\sigma(x_i) : i = 1, \dots, D\}$ of the jumps at the discontinuities x_i of Φ. Moreover, Φ is regular.*

For a subset C of discontinuities of Φ, we denote $\sigma(C) = \sum_{x_i \in C} \sigma(x_i)$ the corresponding sum of jumps of Φ.

Theorem 0.1.4. *Suppose that there are two discontinuities x_{i_0}, x_{j_0} of Φ and a subsequence (q_{n_k}) of denominators of α such that for a constant $\kappa > 0$ we have*

$$q_{n_k} \|(x_{i_0} - x_{j_0}) - r\alpha\| \geq \kappa, \ \forall \ |r| < q_{n_k}. \tag{0.1.5}$$

[13]Let the points $\gamma_{n,\ell}$ run through the set of discontinuities of $\Phi_n := \Phi + \Phi \circ T + \ldots + \Phi \circ T^{n-1}$ in the natural order. The cocycle is said to have *well separated discontinuities (wsd)*, if there are $c > 0$ and an infinite set $\mathcal{Q}$ of denominators of α such that

$$\gamma_{q,\ell+1} - \gamma_{q,\ell} \geq c/q, \ \forall q \in \mathcal{Q}, \, \ell \in \{1, \dots, Dq\}.$$

Then, if the sum $\sigma(C)$ is $\neq 0$ for each non-empty proper subset C of the set of discontinuities of Φ, then Φ has a non-trivial essential value.

In Section 2.4, we present some applications of the results from previous sections to affine cocycles. In particular, in Section 2.4.3., we show that the cocycle (0.1.3) is ergodic for a generic choice (both in the measure-theoretic, as well as topological sense) of parameters $\beta_1, \ldots, \beta_d$, $d \geq 2$.

0.1.10 Description of the content of the dissertation. Chapter 3

Chapter 3 is dedicated to the problem of Markov quasi-similarity and the question of Vershik from Subsection 0.1.3. In Section 3.1 we use the characterization of eigenfunctions proved by Lesigne [32] to show the following result:

Theorem 0.1.5. *A Markov quasi-factor of an automorphism with quasi-discrete spectrum has quasi-discrete spectrum.*

The main result of Section 3.2 (written on the base [14]) is the following theorem:

Theorem 0.1.6. *There exist mixing (of all orders) automorphisms which are MQS but not weakly isomorphic.*

The examples constructed in Section 3.2 are mixing extensions given by Gaussian cocycles of a mixing Gaussian automorphism.

CHAPTER 1

Basic concepts in ergodic theory

1.1 General notions

By a probability standard Borel space one means any probability space $(X, \mathcal{B}, \mu)$ isomorphic to the space $([0,1], \mathcal{B}', \lambda)$, where $\mathcal{B}'$ stands for the σ-algebra of Borel sets and λ denotes Lebesgue measure (sometimes, we may need to complete $\mathcal{B}$).

Equalities between sets, functions, σ-algebras etc. are usually understood modulo null set for the measure μ. If T is a bijection it means an a.e. bijection, i.e. after removing from X a set X_0 of a measure zero, T becomes a genuine bijection from $X \setminus X_0$ onto itself.

By isomorphism one means an a.e. bijection $S : X_1 \to X_2$ which is bimeasurable and which "preserves" the measure, in the sense that $\mu_1(S^{-1}B_2) = \mu_2(B_2)$ for each $B_2 \in \mathcal{B}_2$. Note that $S^{-1} : (X_2, \mathcal{B}_2, \mu_2) \to (X_1, \mathcal{B}_1, \mu_1)$ is also an isomorphism.

1.2 Spectral theory

Let H be a separable Hilbert space and $U : H \to H$ be a unitary operator on H and let $x \in H$.

The sequence $r_n = \langle U^n x, x\rangle, n \in \mathbb{Z}$, is *positive definite*, i.e. for any $(a_n)_{n\geqslant 0} \subset \mathbb{C}$ and $N \geqslant 0$

$$\sum_{n,m=0}^{N} r_{n-m} a_n \overline{a_m} \geqslant 0.$$

It follows from the Herglotz theorem (see e.g. [39]) that any positive definite sequence is the Fourier transform of the unique, finite, non-negative, Borel measure σ_x on $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ [14]. Therefore, we can write

$$\widehat{\sigma}_x[-n] := \int_{\mathbb{S}^1} z^n d\sigma_x = \langle U^n x, x\rangle,\ n \in \mathbb{Z},$$

i.e. $\widehat{\sigma}_x[n]$ denotes the n-th Fourier coefficient of the measure σ_x. The measure σ_x is called the *spectral measure* of x (we may also write $\sigma_{x,U}$ instead of σ_x when a confusion can arise).

Definition 1.2.1. For an element $x \in H$, we define the *cyclic space* $\mathbb{Z}(x)$ generated by x:

$$\mathbb{Z}(x) = \overline{span}\{U^n x :\ n \in \mathbb{Z}\}.$$

It is the smallest, closed, invariant subspace containing x.

Definition 1.2.2. A decomposition $H = \bigoplus_{i=1}^{\infty} \mathbb{Z}(x_i)$ is called *spectral* if $\sigma_{x_1} \gg \sigma_{x_2} \gg \dots$

The main theorem in the spectral theory of unitary operators states that for any unitary operator of a separable Hilbert space a spectral decomposition exists and is unique in the following sense.

Theorem 1.2.1 (Spectral theorem)**.** *If $U : H \to H$ is unitary and $H = \bigoplus_{i=1}^{\infty} \mathbb{Z}(x_i) = \bigoplus_{i=1}^{\infty} \mathbb{Z}(x_i')$ are two spectral decompositions of H then $\sigma_{x_i} \equiv \sigma_{x_i'}$ for every $i \geqslant 1$.*

Definition 1.2.3. The type[15] of the measure σ_{x_1} is called the *maximal spectral type σ_U* of U.

Let $A_1 = \mathbb{S}^1$ and for $n \geqslant 2$ denote

$$A_n = supp\frac{d\sigma_{x_n}}{d\sigma_{x_1}}.^{16} \tag{1.2.1}$$

[14] It is also known that, for each $x, y \in H$, there exists (a unique) complex measure $\sigma_{x,y}$ such that $\widehat{\sigma}_{x,y}[-n] = \langle U^n x, y\rangle$ for every $n \in \mathbb{Z}$. Moreover, $\sigma_{x,y} \ll \sigma_x$ for all x, y.

[15] By the type of a measure one means the set of equivalent measures to a given one. In what follows, if no confusion arises we will not distinguish a measure and its type.

[16] The symbol $\frac{d\mu}{d\nu}$ stands for Radon-Nikodym derivative of μ with respect to ν, where $\mu \ll \nu$.

The sets A_n are defined σ_{x_1}-a.e. and moreover

$$\frac{d\sigma_{x_{n+1}}}{d\sigma_{x_1}} = \frac{d\sigma_{x_{n+1}}}{d\sigma_{x_n}}\frac{d\sigma_{x_n}}{d\sigma_{x_1}}.$$

Thus,

$$\mathbb{S}^1 = A_1 \supset A_2 \supset A_3 \supset \ldots \text{(modulo } \sigma_{x_1}).$$

Definition 1.2.4. The function $M_U : \mathbb{S}^1 \to \{1, 2, \ldots\} \cup \{+\infty\}$ defined by

$$M_U(z) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(z)$$

is called the *multiplicity function* of U.

Definition 1.2.5. Two unitary operators U_i on H_i, $i = 1, 2$, are called *spectrally equivalent* if there exists a surjective isometry $W : H_1 \to H_2$ such that $W \circ U_1 = U_2 \circ W$.

Theorem 1.2.2. *Unitary operators U_i on H_i, $i = 1, 2$, are spectrally equivalent if and only of $\sigma_{U_1} = \sigma_{U_2}$ and $M_{U_1} = M_{U_2}$ (σ_{U_1}-a.e.).*

It follows that the maximal type of U and the multiplicity function determine the operator U.

Definition 1.2.6. Depending on the form of M_U and σ_U one says that:

- U has *simple* spectrum, if $M_U \equiv 1$,
- U has *spectrum of uniform multiplicity* N, if $M_U \equiv N$,
- U has *singular* spectrum, if σ_U is singular with respect to Lebesgue measure,
- U has *absolutely continuous (Lebesgue) spectrum*, if σ_U is absolutely continuous with (equivalent to) Lebesgue measure,
- U has *discrete* spectrum, if σ_U is a discrete measure [17].

It is well-known that the operator $U : \mathbb{Z}(x) \to \mathbb{Z}(x)$ is spectrally equivalent to the operator $V_{\sigma_x} : L^2(\mathbb{S}^1, \sigma_x) \to L^2(\mathbb{S}^1, \sigma_x)$ given by $V_{\sigma_x}(f)(z) = zf(z)$.

We will make use of the following well-known lemmas.

[17] When $U = U_T$ then it is easy to see that σ_U is discrete if and only if T has discrete spectrum in the sense of Definition 1.9.2.

Lemma 1.2.3. *Let U_i be a unitary operator of a separable Hilbert space H_i, $i = 1, 2$. Let $V : H_1 \to H_2$ be linear and continuous operator intertwining U_1 and U_2. Then $\sigma_{Vx,U_2} \ll \sigma_{x,U_1}$.*

Proof. For each $n \in \mathbb{Z}$, we have

$$\langle U_2^n Vx, Vx\rangle = \langle VU_1^n x, Vx\rangle = \langle U_1^n x, V^*Vx\rangle.$$

Therefore, $\widehat{\sigma}_{Vx}[n] = \widehat{\sigma}_{x,V^*Vx}[n]$ for each $n \in \mathbb{Z}$ (cf. footnote 14). Since $\sigma_{x,V^*Vx} \ll \sigma_x$, the assertion follows. □

Lemma 1.2.4. *Let U_1, U_2, V be as above. If $\overline{ImV} = H_2$ then the set of eigenvalues of U_2 is a subset of the set of eigenvalues of U_1.*

Proof. Let $H_i = H_{id} \oplus H_{ic}$ be the decomposition of H_i into two invariant, closed subspaces, where $\sigma_{U_i|H_{id}}$ is discrete and $\sigma_{U_i|H_{ic}}$ is continuous, $i = 1, 2$. By Lemma 1.2.3, $V(H_{1d}) \subset H_{2d}$, $V(H_{1c}) \subset H_{2c}$. Suppose that $U_2 y = cy$ and c in not an eigenvalue of U_1. Then $y \perp V(H_{1d})$ and $y \perp H_{2c}$, whence $y \perp V(H_1)$ and therefore $y = 0$, since $\overline{VH_1} = H_2$. □

For more information on the subject, see e.g. [11], [19], [26].

1.3 Ergodicity, mixing, rigidity

Definition 1.3.1. One says that $T \in Aut(X, \mathcal{B}, \mu)$ is:

- *ergodic*, if for every $A \in \mathcal{B}$ [18]

  $$T^{-1}A = A \;\Rightarrow (\mu(A) = 0 \text{ or } \mu(A) = 1);$$

  equivalently,

  $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mu(T^k A \cap B) = \mu(A)\mu(B)$$

  for all $A, B \in \mathcal{B}$,

- *totally ergodic*, if T^k is ergodic for every $k > 0$,

[18] We also consider infinite measure-preserving automorphisms, e.g. $T_\varphi : (X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes m_G) \to (X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes m_G)$, where m_G is a Haar measure of an l.c.s.c. group G which is not compact. In this case, ergodicity means that each invariant set is either of measure zero or its complement set is of measure zero.

- *weakly mixing*, if for all $A, B \in \mathcal{B}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\mu(T^k A \cap B) - \mu(A)\mu(B)| = 0,$$

- *mixing*, if for all $A, B \in \mathcal{B}$

$$\lim_{n\to\infty} \mu(T^n A \cap B) = \mu(A)\mu(B),$$

- *mixing of order k*, for all $A_1, \dots A_k \in \mathcal{B}$

$$\lim_{n_2,\dots,n_k\to\infty, |n_i-n_j|\to\infty, i\neq j} \mu(A_1 \cap T^{n_2} A_2 \cap \dots \cap T^{n_k} A_k) = \prod_{i=1}^{k} \mu(A_i).$$

Definition 1.3.2. An increasing sequence $(q_n) \subset \mathbb{N}$ is called a *rigidity sequence* for T if $T^{q_n} \xrightarrow[n\to\infty]{} Id$ strongly, i.e. $\|U_{T^{q_n}} f - f\| \to 0$ for every $f \in L^2(X, \mathcal{B}, \mu)$.

Some dynamical properties of $T \in Aut(X, \mathcal{B}, \mu)$ have their characterizations in terms of spectral properties of U_T:

- T is ergodic if and only if 1 is a simple eigenvalue of U_T.
- T is totally ergodic if and only if it is ergodic and in the spectrum of U_T there is no non-trivial root of unity.
- T is weakly mixing if and only if U_T has continuous spectrum on the subspace $L^2_0(X, \mathcal{B}, \mu)$ of $L^2(X, \mathcal{B}, \mu)$ of zero mean functions.
- T is mixing if and only if σ_{U_T} on $L^2_0(X, \mathcal{B}, \mu)$ is a Rajchman measure, i.e. $\widehat{\sigma}_{U_T}(n) \xrightarrow[|n|\to\infty]{} 0$.

For more information on the subject, see e.g. [11], [15], [51].

1.4 Induced automorphism and Rokhlin lemma

Let $T \in Aut(X, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ a set of positive measure. Then, for μ-a.e. $x \in A$, the first return time

$$n_A(x) := \inf\{n \geqslant 1 : T^n x \in A\}$$

of x to A is well defined (Poincaré's lemma, e.g. [51]).

The map $T_A : A \to A$ given by $T_A(x) = T^{n_A(x)}(x)$ is called *the first return map* or *induced automorphism*. The induced automorphism is an automorphism of the space $(A, \mathcal{B}_A, \mu_A)$, where $\mathcal{B}_A = \{B \in \mathcal{B} : B \subset A\}$ and $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$ for each $B \in \mathcal{B}_A$. If T is ergodic then T_A is also ergodic.

Definition 1.4.1. T is called *aperiodic* if for each $n \geq 1$, the set $\{x \in X : T^n x = x\}$ has measure zero.

Definition 1.4.2. The collection $A, TA, \ldots T^{h-1}A$ is called a *Rokhlin tower* of the height h and the base A if $T^i A \cap T^j A = \emptyset$ for $0 \leqslant i < j < h$.

Lemma 1.4.1. *Let T be an aperiodic measure-preserving transformation of $(X, \mathcal{B}, \mu)$. Then for arbitrary $h \in \mathbb{N}$ and $\epsilon > 0$ there exists a Rokhlin tower of the height n and the base A such that $\mu\Big(\bigcup_{k=0}^{h-1} T^k A\Big) > 1 - \epsilon$.*

Remark 1.4.1. It is well known that if we fix $\epsilon >$, $h \geq 1$ and a set $A \in \mathcal{B}$ of positive measure then we can find a Rokhlin tower $F, \ldots, T^{h'-1}F$ with $h' \geq h$, $\mu(\bigcup_{i=0}^{h'-1} T^i F) > 1 - \epsilon$ and such that $F \subset A$.

Remark 1.4.2. It follows directly from definition that if $F, TF, \ldots, T^{h-1}F$ is a Rokhlin tower and $A = T^k F$ (for some $0 \leqslant k \leqslant h-1$) then $n_A(x) \geqslant h$.

For more information on the subject, see e.g. [11], [15], [51].

1.5 Factors and extensions

Assume that $T \in Aut(X, \mathcal{B}, \mu)$, $S \in Aut(Y, \mathcal{C}, \nu)$.

Recall that S is said to be a *factor* of T (and T to be an *extension* of S) if there exists a "measure-preserving" map $R : X \to Y$ [19] such that $R \circ T = S \circ R$ [20].

Notice that if S is a factor of T (with R as above) then $R^{-1}(\mathcal{C})$ is a T-invariant sub-σ-algebra of $\mathcal{B}$. On the other hand, with every invariant sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$ we can associate a factor of T[21].

Assuming T is ergodic, it was shown in [3] that every ergodic extension $\widetilde{T}$ of T can be realized as a *skew product* over T, i.e. the automorphism $\widetilde{T} \in Aut(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ is isomorphic with some automorphism $\overline{T} : (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu) \to (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ given by

$$\overline{T}(x, y) = (Tx, S_x y), \tag{1.5.1}$$

[19] Recall that it means that $\nu = R_*(\mu)$, i.e. for every $A \in \mathcal{C}$, $\nu(A) = \mu(R^{-1}A)$.

[20] If $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu)$ then each measure-preserving $R : X \to Y$ is called an *endomorphism*.

[21] One defines the factor as: $Y = X/\mathcal{A}$ (we do not distinguish points which cannot be separated by the sets from $\mathcal{A}$), the measure structure is given by $(\mathcal{A}, \mu|_{\mathcal{A}})$ and T operates in a natural way on the resulting quotient space. Sometimes, we will denote the resulting factor-automorphism by $T|_{\mathcal{A}}$. We should notice however that to say that factor-automorphisms are defined on probability standard Borel spaces we need a slight extension of the latter notion in General notions and admit atoms.

where $(S_x)_{x\in X} \subset Aut(Y,\mathcal{C},\nu)$ is a measurable family[22]. Note that T is a factor of $\overline{T}$.

A special case of skew products are aforementioned *Rokhlin extensions*, i.e. automorphisms of the form

$$T_{\varphi,\mathcal{G}}(x,y) = (Tx, S_{\varphi(x)}y),$$

where $\mathcal{G} = (S_g)_{g\in G}$ is a measurable representation of an Abelian l.c.s.c. group G in $Aut(Y,\mathcal{C},\nu)$ and $\varphi : X \to G$ is a measurable function (see for example, [29]).

1.6 Coalescence and weak isomorphism

Definition 1.6.1. [36] An automorphism $T \in Aut(X,\mathcal{B},\mu)$ is called *coalescent* if every endomorphism $S : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$, $S \circ T = T \circ S$, is invertible. Equivalently, the operator $U_S f = f \circ S$, $f \in L^2(X,\mathcal{B},\mu)$, is unitary.

It follows from [36] that each automorphism T for which $\sigma_{U_T}(\{\infty\}) = 0$ is coalescent. In particular, all automorphisms with simple spectrum are coalescent.

Later, we will make use of the following well-known observation.

Remark 1.6.1. If T is coalescent and T is weakly isomorphic to T_1, then T and T_1 are isomorphic. Indeed, T_1, as a factor of T, is represented as a T-invariant sub-σ-algebra $\mathcal{B}_1 \subset \mathcal{B}$ (and T_1 can be identified with the quotient action of T on $(X/\mathcal{B}_1, \mathcal{B}_1, \mu|_{\mathcal{B}_1})$). By the same token (reversing the rules of T and T_1) there exists a T-invariant sub-σ-algebra $\mathcal{B}_2 \subset \mathcal{B}_1$ such that the quotient action, say $T|_{\mathcal{B}_2}$, of T on $(X/\mathcal{B}_2, \mathcal{B}_2, \mu|_{\mathcal{B}_2})$ is isomorphic to the original automorphism T. Then, each isomorphism, say S, of T and $T|_{\mathcal{B}_2}$ is in fact an endomorphism of $(X,\mathcal{B},\mu)$ and since $S \circ T = T \circ S$, it must be invertible. Therefore $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$.

1.7 Joinings and Markov operators

Recall that if $T_i \in Aut(X_i,\mathcal{B}_i,\mu_i)$, $i \geqslant 1$, then each $T_1 \times T_2 \times \ldots$-invariant probability measure on $(X_1 \times X_2 \times \ldots, \mathcal{B}_1 \otimes B_2 \otimes \ldots)$ with each X_i-marginal equal to μ_i, is called a *joining* of $T_1, T_2, \ldots$

[22] Here, the measurability means that the map $X \times \mathcal{C} \ni (x,A) \mapsto S_x A \in \mathcal{C}$ is measurable. One considers $\mathcal{C}$ (modulo sets of measure zero) as a metric space with the distance given by the measure of the symmetric difference between sets.

We denote by $J(T_1, T_2, \ldots)$ the set of all joinings of $T_1, T_2, \ldots$ If we assume additionally that each $T_i,\ i \geqslant 1$, is ergodic then $J(T_1, T_2, \ldots)$ is a simplex. Then the extremal points in $J(T_1, T_2, \ldots)$ are exactly ergodic joinings, i.e. all joinings λ for which $(T_1 \times T_2 \times \ldots, \lambda)$ is ergodic. By $J^e(T_1, T_2, \ldots)$ we denote the set of ergodic joinings. For each joining $\lambda \in J(T_1, T_2, \ldots)$, its ergodic decomposition consists of ergodic joinings.

When $T_1 = T_2 = \ldots = T$, we speak about *self-joinings* and we use notation $J_n(T), J_n^e(T)$ if only finitely many, say n, copies of T are involved or $J_\infty(T), J_\infty^e(T)$ for the infinite case. Recall that there is one-to-one correspondence between $J(T_1, T_2)$ and the set of all Markov operators $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ satisfying the intertwining relation $\Phi \circ U_{T_1} = U_{T_2} \circ \Phi$.

Here are the simplest examples of 2-self-joinings of an ergodic $T \in Aut(X, \mathcal{B}, \mu)$:

- the product measure $\mu \otimes \mu$,
- the graph self-joining μ_S corresponding to $S \in C(T)$ [23], where $\mu_S(A \times B) = \mu(A \cap S^{-1}B)$ for each $A, B \in \mathcal{B}$,
- the relative product $\mu \otimes_{\mathcal{A}} \mu$ corresponding to a factor $\mathcal{A} \subset \mathcal{B}$, where $\mu \otimes_{\mathcal{A}} \mu(A \times B) = \int_{X/\mathcal{A}} E(A|\mathcal{A})(\overline{x}) \cdot E(B|\mathcal{A})(\overline{x})\, d(\mu|_{\mathcal{A}})(\overline{x})$.

Note that $\mu_S \in J_2^e(T)$ and the product measure $\mu \otimes \mu \in J_2^e(T)$ if and only if T is weakly mixing. Moreover, the Markov operators corresponding to the three types of joinings listed above are Π [24], $U_S, E(\cdot|\mathcal{A})$, respectively.

For more information about joinings we refer to e.g. [15], [45].

1.8 Essential values of a cocycle taking values in Abelian groups

In this subsection, we recall the definition and general results about essential values of a cocycle (see [1], [46]).

Let $(X, \mathcal{B}, \mu)$ be a (non-atomic) probability standard Borel space and $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ an ergodic automorphism. Such an automorphism is then automatically aperiodic Assume that G is an Abelian l.c.s.c. group with the σ-algebra of its Borel sets $\mathcal{B}(G)$ and a fixed Haar measure m_G (we

[23]By $C(T)$ we denote the *centralizer* of T, i.e. the set of all $R \in Aut(X, \mathcal{B}, \mu), R \circ T = T \circ R$.

[24]The operator $\Pi : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ is given by formula $\Pi f = \int_X f\, d\mu$.

will also write dg instead of m_G). Denote by $\overline{G} = G \cup \{\infty\}$ the one-point compactification of G (when G is non-compact).

For a measurable function $\varphi : X \to G$, we set (φ_n)

$$\varphi_n(x) = \sum_{k=0}^{n-1} \varphi(T^k x),\ n \geq 1,$$

and we extend the formula to all $n \in \mathbb{Z}$ so that, for $n, k \in \mathbb{Z}$, $\varphi_{n+k}(x) = \varphi_n(x) + \varphi_k(T^n x)$. In this way we obtain a cocycle [25] for the $\mathbb{Z}$-action given by $n \mapsto T^n$, $n \in \mathbb{Z}$. For simplicity, the function φ itself will be often called a *cocycle*. We recall that

Definition 1.8.1. A cocycle $\varphi : X \to G$ is called ergodic if the automorphism $T_\varphi : (x, g) \to (Tx, g + \varphi(x))$ is ergodic on $X \times G$ for the measure $\mu \otimes dg$.

1.8.1 Recurrence of a cocycle with values in $\mathbb{R}^d$

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. The inequality $\Big|\|\varphi_{n+1}(x)\| - \|\varphi_n(Tx)\|\Big| \leq \|\varphi(x)\|$ implies the T-invariance of the set $\{x \in X : \lim_n \|\varphi_n(x)\| = +\infty\}$. Therefore, by ergodicity, this set has measure 0 or 1, and we have the following alternative: either for μ-a.e. every $x \in X$, $\lim_n \|\varphi_n(x)\| = +\infty$ or for μ-a.e. $x \in X$, $\liminf_n \|\varphi_n(x)\| < +\infty$.

Definition 1.8.2. A cocycle (φ_n) over (X, μ, T) with values in $G = \mathbb{R}^d$ is called *recurrent* if $\liminf_n \|\varphi_n(x)\| < +\infty$, for a.e. $x \in X$. It is called *transient* if $\lim_n \|\varphi_n(x)\| = +\infty$ for a.e. $x \in X$.

Let us recall some notions and facts from the theory of dynamical systems preserving an infinite measure.

Definition 1.8.3. Let $T \in Aut(X, \mathcal{B}, \mu)$ and μ be a σ-finite measure. A set A is said to be *wandering* if the images $(T^{-n}A)_{n\in\mathbb{Z}}$ are pairwise disjoint. The automorphism T is called *conservative* if there is no wandering measurable set of positive measure.

Lemma 1.8.1. *1. For every $A \in \mathcal{B}$, the set $B := \{x \in A : T^n x \notin A, \forall n \geqslant 1\}$ is a wandering set.*

[25] That is,

$$\varphi_n(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x), & n > 0 \\ 0, & n = 0 \\ -(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)), & n < 0. \end{cases}$$

2. *If F is a set with finite measure and $B \in \mathcal{B}$ a wandering set then for a.e. $x \in B$ the number of visits to F is finite.*

Proof. 1. Suppose $y \in T^{-n_1}B \cap T^{-n_2}B$, with $n_2 > n_1$, then $T^{n_1}y \in B \subset A$ and $T^{n_2-n_1}(T^{n_1}y) \in B \subset A$ which contradicts the definition of B.

2. The observation is based on the Borel-Cantelli type[26] argument. The images of B are pairwise disjoint and therefore we have

$$\sum_{n \geqslant 0} \mu(B \cap T^{-n}F) = \sum_{n \geqslant 0} \mu(T^n B \cap F) \leqslant \mu(F) < +\infty.$$

It follows that for almost every $x \in B$, $\sum_{n \geqslant 0} \mathbb{1}_F(T^n x) < +\infty$.

□

Remark 1.8.1. 1. By 1) of Lemma 1.8.1, conservativity is equivalent to "for every measurable set A, a.e. $x \in A$ returns to A".

2. If $\mu(X) < +\infty$, there is no wandering set of positive measure, hence every dynamical system with finite measure is conservative.

3. In a conservative dynamical system, since a.e. $x \in A$ returns to A at least once, it must return to A infinitely often.

4. If T is conservative then the induced automorphism T_A is conservative for every A of positive measure.

Returning to the theory of cocycles, we obtain the following classical result.

Lemma 1.8.2. *A cocycle (φ_n) is recurrent if and only if T_φ is conservative with respect to $\lambda = \mu \otimes m_G$.*

Proof. For $M > 0$, let $U_M = \{z \in \mathbb{R}^d : ||z|| \leqslant M\}$ and $A_M := X \times U_M$.

"$\Leftarrow$" Suppose that T_φ is conservative. Then a.e. point $(x, z) \in A_M$ returns infinitely many times to A_M. Therefore for a.e. $x \in X$, $\liminf_n ||\varphi_n(x)|| < \infty$.

"$\Rightarrow$" Suppose (φ_n) is recurrent. Let $A \subset X \times \mathbb{R}^d$ be a set of positive measure. We need to show that points of A are returning to A. We can assume that A is bounded, i.e. for some $M > 0$, $A \subset A_M$. Let B be the set of points of A that never return to A. Then B is wandering (see Lemma 1.8.1). Fix $\epsilon > 0$. By recurrence, there exists $L > 0$ such that the set $R := \{x \in X : \liminf_n ||\varphi_n(x)|| < L\}$ has measure μ greater than $1 - \epsilon$. It follows that a.e. $(x, z) \in B \cap (R \times U_M)$:

[26] Borel-Cantelli Lemma : Let $(X, \mathcal{B}, \mu)$ be a probability space and $(C_n)_{n \geqslant 0}$ be a sequence of sets from $\mathcal{B}$. If $\sum_{n=1}^{\infty} \mu(C_n) < \infty$ then for almost all $x \in X$ the set $\{n; x \in C_n\}$ is finite.

- returns to $X \times U_{M+L}$ infinitely often (indeed, $x \in R, ||z|| < M$ and $||\varphi_{n_i}(x)|| < L$ for some $n_1 < n_2 < \dots$; moreover, $T_\varphi^{n_i}(x,z) = (T^{n_i}x, \varphi_{n_i}(x)+z)$ and $||\varphi_{n_i}(x)+z|| \leqslant \|\varphi_{n_i}(x)\| + \|z\| \leqslant L+M$).

- returns to $X \times U_{M+L}$ only finitely many times if we apply Lemma 1.8.1 (for $F = A_{M+L}$),

which yields a contradiction. It follows that $B \subset R^c \times U_M$ (since $B \subset X \times U_M$). Hence $\mu \otimes dg(B) < M\epsilon$ and since $\epsilon > 0$ is arbitrary, $\mu \otimes dg(B) = 0$, and the proof is complete. □

Let us consider $T \in Aut(X, \mathcal{B}, \mu)$, where μ is a finite measure. Let $\varphi : X \to G$ be a cocycle. We induce φ on a set of positive measure A by putting

$$\varphi^A(x) := \varphi_{n_A(x)}(x) = \sum_{j=0}^{n_A(x)-1} \varphi(T^j x).$$

Hence, the "induced" cocycle for the induced automorphism T_A on A is given by

$$\varphi_n^A(x) := \varphi^A(x) + \varphi^A(T_A x) + \ldots + \varphi^A(T_A^{n-1}x), \text{ for } n \geqslant 1.$$

Remark 1.8.2. If (φ_n) is recurrent, then the induced cocycle (φ_n^A) is recurrent. Indeed, $(T_A)_{\varphi_A}$ is the induced automorphism on $A \times G$ of T_φ which is, following the previous lemma, conservative if (φ_n) is recurrent. Clearly, the conservativity of T_φ implies the conservativity of $(T_A)_{\varphi^A}$, hence the recurrence of the cocycle (φ_n^A) over T_A.

The following lemma provides another equivalent condition for recurrence.

Lemma 1.8.3. *A cocycle (φ_n) is recurrent if and only if for each neighborhood $U \ni 0$ and $A \subset X$ of positive measure there exists $n \in \mathbb{Z} \setminus \{0\}$ such that*

$$\mu(A \cap T^{-n}A \cap [\varphi_n \in U]) > 0. \quad (1.8.1)$$

Proof. Following Remark 1.8.2, if (φ_n) is recurrent, then the cocycle (φ_n^A) is recurrent and therefore for every neighborhood U of 0, there is, for a.e. x, an infinite sequence $(n_k) = (n_k(x))$ such that $T^{n_k}x \in A$ and $\varphi_{n_k}(x) \in U$, $\forall k \geq 1$. Let $A_l := \{x \in A : T^l x \in A \text{ and } \varphi_l(x) \in U\}$. Up to a set of measure zero, we have $A = \cup_{l \geqslant 1} A_l$. Therefore, there is $l_0 \geqslant 1$ such that $\mu(A_{l_0}) > 0$, which shows the property (1.8.1).

Conversely, suppose that the cocycle is transient. Then there exists $n_1 \geqslant 1$ such that if we set $Y := \{x \in X : \|\varphi_n(x)\| > 1 \text{ for each } n \geqslant n_1\}$ then $\mu(Y) > 1/2$. Because of the aperiodicity of T, there is $C \subset X$ such that the sets T^kC, for $k = 0, \dots, n_1-1$ are disjoint and $\mu(\cup_0^{n_1-1} T^kC) > 1/2$. It follows that there exists $k \in \{0, 1, \dots, n_1 - 1\}$ such that $\mu(Y \cap T^kC) > 0$. Let $B := T^kC$. The first return time $n_B(x)$ to B of a point $x \in B$ satisfies $n_B(x) \geq n_1$. It follows that for every $n \geq n_1$, we have $\mu(B \cap T^{-n}B \cap [\varphi_n \in U]) = 0$ for $U = \{z \in \mathbb{R}^d : ||z|| < 1\}$. Hence, the property (1.8.1) with U defined above is not satisfied by B since $\mu(B \cap T^{-n}B) = 0$ for $n = 0, \dots, n_1 - 1$. □

Remark 1.8.3. In order to give a simple example of a non-trivial recurrent cocycle consider a cocycle $\varphi : X \to G$ such that $\varphi_{\ell_n} \to 0$ in measure, where (l_n) is a rigidity sequence for T. Then φ is recurrent; indeed, in (1.8.1), $T^{-l_n}A$ is almost equal to A while $[\varphi_{l_n} \in U]$ is almost the whole space X.

Remark 1.8.4. For each $d \geq 1$, the cocycle generated by a (measurable) function $\varphi : \mathbb{T} \to \mathbb{R}^d$ over any irrational rotation, say by α, is recurrent if the components of φ have bounded variation and integral 0. Indeed, by the Denjoy-Koksma inequality (2.2.1), since (φ_{q_n}) is a bounded sequence in $\mathbb{R}^d$ ((q_n) stands for the sequence of denominators of α, see Section 1.8.1), the condition $\liminf_n \|\varphi_n(x)\| < \infty$ holds for every $x \in \mathbb{T}$.

1.8.2 Regular cocycles

Definition 1.8.4. A cocycle $\varphi : X \to G$ is called a *coboundary* if $\varphi = f - f \circ T$ for a measurable map $f : X \to G$. Two cocycles $\varphi, \psi : X \to G$ are called *cohomologous* if $\varphi - \psi$ is a coboundary.

An obvious obstruction to the ergodicity of a cocycle φ is that φ is cohomologous to a cocycle ψ taking its values in a proper closed subgroup of G. This suggests the following definition:

Definition 1.8.5. A cocycle $\varphi : X \to G$ is called *regular* if it is cohomologous to a cocycle ψ taking values in a closed subgroup H of G such that $T_\psi : (x, h) \to (Tx, h + \psi(x))$ is ergodic on $X \times H$ for the measure $\mu \otimes dh$, where dh is a (fixed) it Haar measure on H.

So, a regular cocycle is "almost" ergodic (up to reduction by cohomology to a smaller closed subgroup).

One of the main tools for studying the ergodicity and the regularity of a cocycle is the following notion.

Definition 1.8.6. An element $g \in \overline{G}$ is called an *essential value* for a cocycle φ, if for each open neighborhood $U \ni g$ in $\overline{G}$, for each $A \in \mathcal{B}$ of positive measure, there exists $N \in \mathbb{Z}$ such that $\mu(A \cap T^{-N}A \cap [\varphi_N \in U]) > 0$. We denote the set of essential values by $\overline{\mathcal{E}}(\varphi)$ and we set $\mathcal{E}(\varphi) := \overline{\mathcal{E}}(\varphi) \cap G$.

Note that if $0 \neq g \in \mathcal{E}(\varphi)$ then we have $\mu(A \cap T^{-N}A \cap [\varphi_N \in U]) > 0$ for infinitely many values of $N \in \mathbb{Z}$. Indeed, because T is ergodic and aperiodic, for each $N \in \mathbb{Z} \setminus \{0\}$ we can find a subset $C \subset A$, $\mu(C) > 0$ such that $T^N C \cap C = \emptyset$ (see Remark 1.4.2).

Lemma 1.8.4. *Cocycles with non-trivial essential values are recurrent.*

Proof. Assume that $g \in \mathcal{E}(\varphi) \setminus \{0\}$. We show the property (1.8.1). Take U a neighborhood of $0 \in G$. Then find $N \in \mathbb{Z}$ so that there is $B \subset X$, $\mu(B) > 0$ such that

$$B \subset A,\ T^N B \subset A \text{ and } \varphi_N(B) \subset g + U.$$

Apply once more the definition of the essential value, this time to the set $T^N B$ to find $C \subset X$, $\mu(C) > 0$ and an integer $M \neq N$ such that

$$C \subset T^N B,\ T^M C \subset T^N B \text{ and } \varphi_M(C) \subset g + U.$$

Now, for $x \in C \subset A$ we have $T^{M-N}x = T^{-N}(T^M x) \in T^{-N}(T^N B) = B \subset A$. Moreover,

$$\varphi_{M-N}(x) = \varphi_M(x) + \varphi_{-N}(T^M x) = \varphi_M(x) - \varphi_N(T^{M-N}x) \in U - U$$

since $T^{M-N}x \in B$. □

It turns out that $\mathcal{E}(\varphi)$ is a closed subgroup of G. Besides, two cohomologous cocycles have the same group of essential values.

Let $\sigma_g(x,h) := (x, g+h)$, $g \in G$, be the action of G on $X \times G$ by translations on the second coordinate. Clearly, it commutes with T_φ. Then (see [46], Theorem 5.2) $\mathcal{E}(\varphi)$ is the *stabilizer of the Mackey action* of φ, that is

$$\begin{aligned}\mathcal{E}(\varphi) = \{g \in G : F \circ \sigma_g = F,& \\ \forall \text{ measurable, } T_\varphi\text{-invariant } F : X \times G \to \mathbb{C}\}.&\end{aligned} \tag{1.8.2}$$

In other words, $\mathcal{E}(\varphi)$ is the group of periods of the measurable, T_φ-invariant functions. Therefore, φ is ergodic if and only if $\mathcal{E}(\varphi) = G$. If φ is regular, then the group H in the definition of regularity is necessarily $\mathcal{E}(\varphi)$. Coboundaries are precisely regular cocycles φ with $\mathcal{E}(\varphi) = \{0\}$.

1.9 Examples of dynamical systems

1.9.1 Irrational rotations and continued fraction expansions

Let $\mathbb{T} = [0,1)$ denote the additive circle. Assume that $\alpha \in [0,1)$ is irrational. Set $Tx = x + \alpha \bmod 1$ for $x \in \mathbb{T}$. Moreover, $T \in Aut(\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}})$. Then T

is ergodic and since $U_T(r_n) = e^{2\pi i n\alpha} r_n, r_n(x) = e^{2\pi i n x}$ $(n \in \mathbb{Z})$, T has discrete spectrum (see Definition 1.9.2 below).

Let us recall some basic facts about continued fractions (e.g. [20]). Let $[0; a_1, ..., a_n, ...]$ be the continued fraction expansion of $\alpha \in (0, 1)$, i.e.

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

and let $(p_n/q_n)_{n\geq -1}$ be the sequence of its convergents. The integers p_n (resp. q_n) are the *numerators* (resp. *denominators*) of α. We have $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$, and for $n \geq 1$:

$$p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2},\ (-1)^n = p_{n-1}q_n - p_n q_{n-1}. \quad (1.9.1)$$

As usual, the fractional part of $u \in \mathbb{R}$ is demoted by $\{u\} = u - [u]$, where $[u]$ is the integral part of u. For $u \in \mathbb{R}$, set $\|u\| = \inf_{n\in\mathbb{Z}} |u - n| = \min(\{u\}, 1 - \{u\})$. Then for any integer M we have $\|Mu\| \leqslant |M|\|u\|$. Note that $\|\cdot\|$ yields a translation invariant distance on $\mathbb{T}$.

We have for $n \geq 0$, $\|q_n\alpha\| = (-1)^n\theta_n$ with $\theta_n = q_n\alpha - p_n$, and moreover

$$1 = q_n\|q_{n+1}\alpha\| + q_{n+1}\|q_n\alpha\|, \quad (1.9.2)$$

$$\frac{1}{q_{n+1} + q_n} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}}, \quad (1.9.3)$$

$$\|q_n\alpha\| \leq \|k\alpha\|, \text{ for } 1 \leq |k| < q_{n+1}. \quad (1.9.4)$$

Definition 1.9.1. An irrational α is said to be of *bounded type* if the sequence (a_n) is bounded.

1.9.2 Dynamical systems with discrete and quasi-discrete spectrum

Assume that $T \in Aut(X, \mathcal{B}, \mu)$. Denote by $E_0(T)$ the group all eigenvalues of U_T and for each integer $k > 0$ set

$$E_k(T) = \{f \in L^2(X, \mathcal{B}, \mu) : |f| = 1, f \circ T \cdot \overline{f} \in E_{k-1}(T)\}.$$

Remark 1.9.1. Observe that when T is ergodic, the set $\{\lambda f : \lambda \in \mathbb{C}$ and $f \in E_1(T)\}$ is the set of eigenfunctions for U_T.

Definition 1.9.2. T is said to have a *discrete spectrum* if $E_1(T)$ is linearly dense in $L^2(X, \mathcal{B}, \mu)$.

Definition 1.9.3. T is said to have *quasi-discrete spectrum* if T is totally ergodic and the subspace generated by $\cup_{k\in\mathbb{N}}E_k(T)$ equals $L^2(X,\mathcal{B},\mu)$.

Clearly, each irrational rotation $Tx = x+\alpha$ has quasi-discrete spectrum. The simplest example of an automorphism with quasi-discrete but not discrete spectrum is given by the group extension T_φ of T by an affine cocycle $\varphi(x) = mx + c$ ($\varphi : \mathbb{T} \to \mathbb{T}$, $m \in \mathbb{Z}\setminus\{0\}$, $c \in \mathbb{T}$). That is, $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$, $T_\varphi(x,y) = (x+\alpha, mx+y+c)$. Indeed, for $q_k(x,y) = e^{2\pi iky}$ we have $q_k \circ T_\varphi(x,y) = e^{2\pi ikc}r_{km}(x)q_k(x,y)$, so $q_k \in E_2(T_\varphi)$.

Remark 1.9.2. Assume that T has quasi-discrete spectrum and let $\rho \in J^e_\infty(T)$. Then $\bigcup_{k\geqslant 0}^{\infty} E_k(T\times T\times\ldots,\rho)$ is linearly dense in $L^2(X\times X\times\ldots,\rho)$. Indeed, by induction on k we show that whenever $f_1,\ldots,f_r \in E_k(T)$ and we let $F(x_1,x_2,\ldots) := f_1(x_1)\cdot f_2(x_2)\ldots\cdot f_r(x_r)$, then $F\circ(T\times T\times\ldots)\cdot\overline{F} \in E_{k-1}(T\times T\times\ldots,\rho)$, so $F\in E_k(T\times T\times\ldots,\rho)$.

It looks as if ergodic self-joinings of an automorphism with quasi-discrete spectrum yield automorphisms with quasi-discrete spectrum. However, this is not the case because an ergodic self-joining need not be totally ergodic.[27] We will bypass such a difficulty in Chapter 3.

We define $G_n(T) = \{f : X \to \mathbb{S}^1 : f \in L^2(X,\mathcal{B},\mu),\ R^n f = 1\}$ for $n = 0,1,2,\ldots$, where $Rf = f\circ T/f$ ($R^0 = Id$).

Lemma 1.9.1. *Let T be ergodic, then $G_n(T) = E_{n-1}(T)$ for $n \geqslant 2$.*

Proof. Induction on n. Let $n=2$ and assume that $f \in L^2(X,\mathcal{B},\mu)$, $|f| = 1$ and $R^2 f = 1$. It follows that $\frac{f\circ T}{f}\circ T = \frac{f\circ T}{f}$, that is $\frac{f\circ T}{f}$ is T-invariant, whence by the ergodicity of T, it is constant. Thus $G_2(T)\subset E_1(T)$. Obviously, $E_1(T)\subset G_2(T)$.

Assume now that $G_n(T) = E_{n-1}(T)$ and take $f\in G_{n+1}(T)$. It follows that $R^{n+1}(f) = R^n(Rf) = 1$ and therefore $Rf \in G_n(T) = E_{n-1}(T)$. By the definitions of $E_{n-1}(T)$ and R it follows that $f\circ T = g\cdot f$ with $g\in E_{n-1}(T)$ which implies $f \in E_n(T)$.

Conversely, if $f\in E_n(T)$ then $f\circ T = g\cdot f$ for $g \in E_{n-1}(T) = G_n(T)$ (the latter equality follows by the induction assumption). It follows that $R^{n+1}f = R^n(Rf) = R^n(g) = 1$ and therefore $f\in G_{n+1}(T)$. □

[27] Indeed, if $\overline{T}(x,y) = (x+\alpha, x+y)$ then the automorphism $(x,y,z) \overset{W}{\longmapsto} (x+\alpha, y+z, z+x+1/2)$ is a self-joining of $\overline{T}$. Moreover, $F(x,y,z) = e^{2\pi i(z-y)}$ satisfies $F\circ W = -F$. However, W is not ergodic (see Subsection 1.9.4 and consider the character $(y,z)\mapsto 2y-2z$). On the other hand, for a.e. ergodic component of W, F restricted to it satisfies the equation above. Moreover, for some of these ergodic components $F\neq 0$. An ergodic component of W is still an ergodic self-joining of $\overline{T}$ (cf. Subsection 1.7).

Lemma 1.9.2. *Let T be ergodic and S an endomorphism of $(X, \mathcal{B}, \mu)$ such that $S \circ T = T \circ S$. Then $U_S(G_n(T)) \subset G_n(T)$.*

Proof. Induction on n. Clearly, the assertion is true for $n = 0, 1$ as $G_0 = \{1\}$ and $G_1 = \{c : c \in \mathbb{S}^1\}$. In view of Lemma 1.9.1, we only need to show (inductively) that $U_S(E_k(T)) \subset E_k(T)$ for $k \geq 1$. Assume that $f \in E_k(T)$. Then $f \circ T = g \cdot f$ with $g \in E_{k-1}(T)$. We have

$$(f \circ S) \circ T = (f \circ T) \circ S = (g \cdot f) \circ S = g \circ S \cdot f \circ S.$$

By the inductive assumption, $g \circ S \in E_{k-1}(T)$ and therefore $f \circ S \in E_k(T)$. □

Lemma 1.9.3. *If $g \in G_{n+1}(T)$ and S an endomorphism such that $S \circ T = T \circ S$ then $g \circ S/g \in G_n(T)$.*

Proof. For $n = 0$ the assertion is true as the ergodicity of T implies that g is constant whenever $g \in G_1(T)$. Thus $g \circ S/g = 1 \in G_0(T)$. Suppose now that $g \circ S/g \in G_n(T)$ whenever $g \in G_{n+1}(T)$ for $n = 1, 2, \ldots, m$. Let $g \in G_{m+2}(T)$. Then $g \circ T = h \cdot g$ with $h \in G_{m+1}(T)$. Thus, $g \circ S \circ T = h \circ S \cdot g \circ S$ and $g \circ S \circ T/g \circ T = (h \circ S/h)(g \circ S/g)$, where, by assumption, $h \circ S/h \in G_m(T)$. By the definition of $G_{m+1}(T)$, we have $g \circ S/g \in G_{m+1}(T)$. □

We will make use of the following theorem.

Theorem 1.9.4. *([17]) If T has quasi-discrete spectrum, then T is coalescent.*

Proof. Assume that S is an endomorphism of $(X, \mathcal{B}, \mu)$ which commutes with T. We need to show that U_S is unitary. By Lemma 1.9.2, it suffices to show that U_S maps $G_n(T)$ onto itself for $n = 0, 1, \ldots$ Clearly, it is true for $n = 0$. Suppose now that U_S maps $G_n(T)$ onto $G_n(T)$. We have to show that U_S maps $G_{n+1}(T)$ onto $G_{n+1}(T)$. By Lemma 1.9.3, if $g \in G_{n+1}(T)$, then $g \circ S/g = h \in G_n(T)$. By the induction assumption, there exists $h' \in G_n(T)$ for which $U_S h' = h' \circ S = h$. If we now set $g' := g/h$ then we have $U_S g' = g' \circ S = g \circ S/h' \circ S = g \circ S/h = g$.

□

We also recall the following classical result by Hahn and Parry ([17]):

Theorem 1.9.5. *If T has quasi-discrete spectrum, S is a factor of it, then S has quasi-discrete spectrum.*

For a full classification of automorphisms with quasi-discrete spectrum see [2] [28].

[28] Each automorphism T with quasi-discrete spectrum can be represented as $Tx = Ax + b$ where X is an Abelian compact connected and metrizable group, A is (continuous, group)

1.9.3 Gaussian dynamical systems

We will recall now necessary facts from [24] and [25] needed for the sequel.

Assume that σ is a finite, continuous, symmetric, Borel measure on $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then, on the space $X_\sigma = \mathbb{R}^{\mathbb{Z}}$ endowed with the natural Borel structure there exists a probability measure μ_σ (called a Gaussian measure) such that the process $(P_n)_{n\in\mathbb{Z}}$ defined by

$$P_n : X_\sigma \to \mathbb{R}, \quad P_n(\omega) = \omega_n \quad \text{for} \quad n \in \mathbb{Z},$$

is a real stationary centered Gaussian process whose spectral measure is σ, i.e.

$$\widehat{\sigma}[-n] = \int_{\mathbb{S}^1} z^n \, d\sigma(z) = \int_{X_\sigma} P_n P_0 \, d\mu_\sigma \quad \text{for all} \quad n \in \mathbb{Z}.$$

If we denote by T_σ the shift transformation[29] on X_σ then the automorphism $T_\sigma : (X_\sigma, \mu_\sigma) \to (X_\sigma, \mu_\sigma)$ is a (standard) Gaussian automorphism [30] with the real Gaussian space

$$H_\sigma = \overline{\text{span}}\{P_n = P_0 \circ T_\sigma^n : n \in \mathbb{Z}\} \subset L^2(X_\sigma, \mu_\sigma).$$

The space H_σ corresponds to the subspace $\mathscr{H}_\sigma$ of $L^2(\mathbb{S}^1, \sigma)$ consisting of functions g satisfying $g(\overline{z}) = \overline{g(z)}$ (σ-a.e). In this representation, the action of U_{T_σ} on $\mathscr{H}_\sigma$ is given by $V(g)(z) = zg(z)$, while the variable P_0 corresponds to the constant function $\mathbb{1} = \mathbb{1}_{\mathbb{S}^1}$. If $g \in \mathscr{H}_\sigma \subset L^2(\mathbb{S}^1, \sigma)$ is of modulus 1 (a.e.), then it determines a unitary operator W on $L^2(\mathbb{S}^1, \sigma)$ acting by the formula $W(f)(z) = g(z)f(z)$. Moreover, $W \circ V = V \circ W$. Then, there is a unique extension of W to a unitary (Koopman) operator U_S on $L^2(X_\sigma, \mu_\sigma)$, where $S : (X_\sigma, \mu_\sigma) \to (X_\sigma, \mu_\sigma)$ and S belongs to the Gaussian centralizer $C^g(T_\sigma)$ of T_σ i.e. to the set of all elements of the centralizer $C(T_\sigma)$ which preserve the Gaussian space H_σ (note that $C^g(T_\sigma)$ is Abelian). Because of the continuity of σ, T_σ is ergodic, in fact, weakly mixing. We also recall that if T_σ is mixing (equivalently, σ is a Rajchman measure, i.e. $\widehat{\sigma}[n] \xrightarrow[|n|\to\infty]{} 0$) then it is mixing of all orders (Leonov's theorem, for a simple proof see e.g. [24]).

automorphism of X and $b \in A$. Moreover, it has zero entropy. Note that $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$, $T_\varphi(x, y) = (x + \alpha, mx + y + c)$ is of the above form with A given by the matrix $\begin{bmatrix} 1 & 0 \\ m & 0 \end{bmatrix}$ and $b = (\alpha, c)$.

[29] $T_\sigma((w_n)_{n\in\mathbb{Z}}) = (v_n)_{n\in\mathbb{Z}}$, where $v_n = w_{n+1}$, $n \in \mathbb{Z}$.

[30] Assume that $T \in Aut(X, \mathcal{B}, \mu)$. T is said to be a *Gaussian automorphism* if: (i) there exists a real U_T-invariant closed subspace $H \subset L^2(X, \mathcal{B}, \mu)$ such that each $0 \neq h \in H$ is a Gaussian variable, (ii) the smallest σ-algebra $\mathcal{B}(H) \subset \mathcal{B}$ with the respect to which all $h \in H$ are measurable is equal to $\mathcal{B}$. H is then called the Gaussian space of T.

Following [25], T_σ is called GAG (or σ is a GAG measure) if for each $\rho \in J_2^e(T_\sigma)$ we have all non-zero variables $(\omega, \omega') \mapsto Q(\omega) + Q'(\omega')$ Gaussian whenever $Q, Q' \in H_\sigma$, i.e. the automorphism $(T_\sigma \times T_\sigma, \rho)$ is Gaussian (with the Gaussian space equal to $\overline{H_\sigma + H_\sigma}$). All Gaussian automorphisms with simple spectrum are GAG and $C(T_\sigma) = C^g(T_\sigma)$ (see [25]) .

Each variable $Q \in H_\sigma$ ($Q : X_\sigma \to \mathbb{R}$) can be treated as a real-valued cocycle (for T_σ). It is called a (real) *Gaussian cocycle*. A Gaussian cocycle Q is called a *Gaussian coboundary* if it is a coboundary $Q = J - J \circ T_\sigma$ with $J \in H_\sigma$.

Remark 1.9.3. Note that it means that if $f \in \mathscr{H}_\sigma$ corresponds to Q, then $f(z) = \xi(z) - V(\xi)(z) = \xi(z)(1-z)$ for some $\xi \in L^2(\mathbb{S}^1, \sigma)$; equivalently $f(z)/(1-z) \in L^2(\mathbb{S}^1, \sigma)$. In this case f is called an $L^2(\mathbb{S}^1, \sigma)$*-coboundary.*

The following result has been proved in [24].

Proposition 1.9.6 ([24])**.** *Assume that $Q \in H_\sigma$. Then the following conditions are equivalent:*

(i) $Q : X_\sigma \to \mathbb{R}$ is a coboundary;

(ii) $Q : X_\sigma \to \mathbb{R}$ is a Gaussian coboundary;

(iii) $e^{2\pi i Q} : X_\sigma \to \mathbb{S}^1$ is a coboundary;

(iv) there exists $|c| = 1$ such that $e^{2\pi i Q} = c \cdot \xi/\xi \circ T$ for some measurable $\xi : X_\sigma \to \mathbb{S}^1$.

1.9.4 Abelian compact group extensions of GAG automorphisms

Assume that $T = T_\sigma$ is a GAG. It then acts on the space $(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma)$. Let G be a compact (metric) Abelian group and let $\varphi : X_\sigma \to G$ be a cocycle. We recall that ([41], Theorem 4.8):

φ is ergodic if and only if for no $1 \neq \chi \in \widehat{G}$ there is a measurable solution $\zeta : X_\sigma \to \mathbb{S}^1$ to the functional equation

$$\chi \circ \varphi = \zeta \circ T/\zeta. \tag{1.9.5}$$

Moreover, when T_φ is ergodic then $c \in \mathbb{S}^1$ is an eigenvalue of T_φ if and only if there are $\chi \in \widehat{G}$ and $\zeta : X_\sigma \to \mathbb{S}^1$ measurable

$$\chi \circ \varphi = c \cdot \zeta \circ T/\zeta. \tag{1.9.6}$$

Corollary 1.9.7. *If T is a GAG and $Q : X_\sigma \to \mathbb{R}$ is a Gaussian cocycle then $T_{e^{2\pi i Q}}$ is ergodic if and only if it is weakly mixing.*

Proof. The result follows directly from (1.9.6) and Proposition 1.9.6 (see (iv)).

□

Assume that $\varphi : X \to G$ is an ergodic cocycle. Then it follows from [25], [36] that $T = T_\sigma$ is a canonical factor of T_φ, that is, if $\mathcal{A} \subset \mathcal{B}_\sigma \otimes \mathcal{B}(G)$ is a factor of T_φ such that $T_\varphi|_{\mathcal{A}}$ is isomorphic to T then $\mathcal{A} = \mathcal{B}_\sigma \otimes \{\emptyset, G\}$. It follows that if $\widetilde{W}$ is an endomorphism commuting with T_φ then it has to preserve $\mathcal{B}_\sigma \otimes \{\emptyset, G\}$ and moreover, by [36],

$$\widetilde{W} = W_{f,V}, \quad W_{f,V}(\omega, g) = (W\omega, f(\omega) + V(g)), \tag{1.9.7}$$

where $W \in C^g(T)$, $f : X_\sigma \to G$ is measurable, and $V : G \to G$ is a continuous epimorphism. Notice that

$$W_{f,V} \circ T_\varphi = T_\varphi \circ W_{f,V}$$

implies

$$V\varphi(\omega) - \varphi(W\omega) = f(\omega) - f(T\omega). \tag{1.9.8}$$

Remark 1.9.4. Recall also that Rudolph in [44] proved that if $S \in Aut(Y, \mathcal{C}, \nu)$ is mixing, $\psi : Y \to G$ (G is compact (metric) Abelian group) is a cocycle such that S_ψ is weakly mixing, then S_ψ is mixing. In particular, if T is a mixing GAG and Q is a Gaussian cocycle which is not a coboundary then $T_{e^{2\pi i Q}}$ is mixing. The above result also hold for k-fold mixing.

CHAPTER 2

Strong regularity of affine cocycles

2.1 More about cocycles

2.1.1 Essential values of cocycles taking values in Abelian groups

The following lemmas show how essential values and regularity behave when a group homomorphism is applied to a cocycle.

Lemma 2.1.1. *Assume that $\varphi : X \to G$ is a cocycle and let $M : G \to H$ be a (continuous) group homomorphism. Then $M\mathcal{E}(\varphi) \subset \mathcal{E}(M\varphi)$. If M is an isomorphism, then $M\mathcal{E}(\varphi) = \mathcal{E}(M\varphi)$.*

Proof. Let $p \in \mathcal{E}(\varphi)$. We want to show that Mp is a period of the measurable $T_{M\varphi}$-invariant functions on $X \times H$. Let $F : X \times H \to \mathbb{C}$ be such a function. Moreover, by a standard argument, we can modify F on a set of zero measure in order to obtain a function (still denoted by F) which is $T_{M\varphi}$-invariant everywhere.

Let us fix $h \in H$ and denote $F_h : X \times G \to \mathbb{C}$ by setting $F_h(x, y) = F(x, h + My)$. We have

$$\begin{aligned}(F_h \circ T_\varphi)(x, y) &= F_h(Tx, y + \varphi(x)) = F(Tx, h + My + M\varphi(x)) \\ &= F(x, h + My) = F_h(x, y).\end{aligned}$$

In view of (1.8.2), $p \in \mathcal{E}(\varphi)$ is a period for F_h, i.e., $F_h(x, y + p) = F_h(x, y)$ for a.e. (x, y). This implies that, for every $h \in H$ and for a.e. (x, y),

$F(x, h + My + Mp) = F(x, h + My)$. By Fubini, this implies that there is $y \in G$ such that for a.e. (x, h), $F(x, h + My + Mp) = F(x, h + My)$.

By invariance of the Haar measure, this implies $F(x, h + Mp) = F(x, h)$, for a.e. (x, y) and Mp is a period of F.

For the second part of the assertion, apply the above to $M\varphi$ and M^{-1}. □

We have the following lemma (cf. Lemma 2.9 in [8]):

Lemma 2.1.2. *If φ is a cocycle on (X, μ, T) with values in an Abelian l.c.s.c. group G and H a closed subgroup of G, then the subgroup $\mathcal{E}(\varphi)/H$ of G/H is such that*

$$\mathcal{E}(\varphi)/H \subset \mathcal{E}(\varphi + H). \tag{2.1.1}$$

If $H \subset \mathcal{E}(\varphi)$, then we have

$$\mathcal{E}(\varphi)/H = \mathcal{E}(\varphi + H). \tag{2.1.2}$$

Moreover, $\varphi^* := \varphi + H : X \to G/H$ is regular if and only if φ is regular.

Proof. Whenever $H \subset G$ is a closed subgroup, (2.1.1) follows from Lemma 2.1.1 applied to the homomorphism $g \in G \to g + H \in G/H$.

Now suppose that $H \subset \mathcal{E}(\varphi)$. In view of (2.1.1) it remains to show that $\mathcal{E}(\varphi + H) \subset \mathcal{E}(\varphi)/H$. Take $g_0 + H \in \mathcal{E}(\varphi + H)$. All we need to show is that there exists $h_0 \in H$ such that $g_0 + h_0 \in \mathcal{E}(\varphi)$, which, by $H \subset \mathcal{E}(\varphi)$, is equivalent to showing that $g_0 \in \mathcal{E}(\varphi)$.

Take $F : X \times G \to \mathbb{C}$ which is measurable and T_φ-invariant. Since $H \subset \mathcal{E}(\varphi)$, $F \circ \sigma_h = F$ for each $h \in H$ because of (1.8.2). We can defined $\tilde{F}$ on $X \times G/H$ such that $\tilde{F}(x, g + H) = F(x, g)$. Since $g_0 + H \in \mathcal{E}(\varphi + H)$, again using (1.8.2), we obtain that $\tilde{F} \circ \sigma_{g_0+H} = \tilde{F}$, which by H-invariance of F means $F \circ \sigma_{g_0} = F$ and therefore $g_0 \in \mathcal{E}(\varphi)$

Assume now that φ^* is regular. So there are a measurable $\eta^* : X \to G/H$ and a closed subgroup $J^* \subset G/H$ such that

$$\psi^*(x) := \varphi^*(x) + \eta^*(x) - \eta^*(Tx) \in J^* \subset G/H$$

and T_{ψ^*} is ergodic on $X \times J^*$, i.e. $\mathcal{E}(\psi^*) = J^*$. Let $\pi : G \to G/H$ be the canonical homomorphism and $s : G/H \to G$ a measurable selector, that is, $s(g + H) \in g + H$ for each $g + H \in G/H$. Then $J := \pi^{-1}(J^*)$ is a closed subgroup of G. Denote $\eta := s \circ \eta^*$ and set

$$\varphi'(x) := \varphi(x) + \eta(x) - \eta(Tx).$$

Then $\varphi'(x)+H=\varphi^*(x)+\eta^*(x)-\eta^*(Tx)=\psi^*(x)\in J^*$, whence $\varphi':X\to J$. By (3.2.10), since $\mathcal{E}(\varphi')=\mathcal{E}(\varphi)$, we have

$$\mathcal{E}(\varphi')/H=\mathcal{E}(\varphi)/H=\mathcal{E}(\varphi+H)=\mathcal{E}(\varphi^*)=J^*,$$

so $\mathcal{E}(\varphi')=J$ and φ is regular.

Conversely, if φ is regular then $\varphi=\eta-\eta\circ T+\psi$, where $\eta:X\to G$ is measurable and $\psi:X\to\mathcal{E}(\varphi)$. Then φ^* is cohomologous to $\psi+H$ which takes values in $\mathcal{E}(\psi)/H=\mathcal{E}(\varphi)/H=\mathcal{E}(\varphi+H)$ by (3.2.10), so φ^* is regular. □

A particular case is when $H=\mathcal{E}(\varphi)$. For $\varphi^*=\varphi+\mathcal{E}(\varphi)$, we get: $\mathcal{E}(\varphi^*)=\{0\}$ and φ is regular if and only if φ^* is regular (hence a coboundary).

It can be shown that a cocycle φ is a coboundary if and only if $\overline{\mathcal{E}}(\varphi)=\{0\}$. This includes in particular the fact that, if φ has its values in a compact group and has no non trivial essential values, it is a coboundary.

Hence *regularity is equivalent to* $\overline{\mathcal{E}}(\varphi^*)=\{0\}$. *In particular cocycles wi values in compact groups, or more generally such that* $\mathcal{E}(\varphi)$ *has a compact quotient in* G*, are regular.*

Lemma 2.1.3. *Assume that* $\varphi:X\to G$ *is a cocycle and let* $M:G\to H$ *be a (continuous) group homomorphism. If* $\varphi:X\to G$ *is regular, so is* $M\varphi:X\to H$.

Proof. If φ is regular, there is a cocycle $\psi:X\to J$ with values in a closed subgroup $J\subset G$ and a measurable function $f:X\to G$ such that

$$\varphi=f-f\circ T+\psi$$

and $T_\psi:(x,j)\to(Tx,j+\psi(x))$ is ergodic on $X\times J$. Thus $M\varphi=Mf-(Mf)\circ T+M\psi$.

We have $\mathcal{E}(\psi)=J$ by ergodicity of T_ψ on $X\times J$ and $MJ=M\mathcal{E}(\psi)\subset\mathcal{E}(M\psi)$ by Lemma 2.1.1. Since $M\psi:X\to MJ\subset\overline{MJ}$, it implies $\mathcal{E}(M\psi)\subset\overline{MJ}$. But $\mathcal{E}(M\psi)$ includes MJ and is closed, so it is equal to $\overline{MJ}$.

Hence $T_{M\psi}$ is ergodic on $X\times\overline{MJ}$, which implies the regularity of $M\varphi$. □

The lemma gives a variant of the proof of the second part of Lemma 2.1.2. It shows that if φ has a non regular quotient then it is non regular.

Remark 2.1.1. Assume that $\psi:X\to G_1\times G_2$ is a cocycle of the form $\psi=(0,\psi_2)$ with $\psi_2:X\to G_2$. Then $\mathcal{E}(\psi)=\{0\}\times\mathcal{E}(\psi_2)$. Indeed, $\psi_N(x)$ is close to (g_1,g_2) if and only if g_1 is close to zero and $(\psi_2)_N(x)$ is close to g_2, so this equality follows directly from the definition of essential value. Moreover, clearly ψ is a regular cocycle if ψ_2 is regular and the converse follows from Lemma 2.1.3.

Finally we recall some effective tools which can be used to find essential values of a cocycle. Given $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and $\varphi : X \to G$, we denote the image of μ on G via φ by $\varphi_*\mu$. We will make use of the following essential value criterion.

Proposition 2.1.4 ([30])**.** *Assume that T is ergodic and let $\varphi : X \to G$ be a cocycle with values in an Abelian l.c.s.c. group G. Let (ℓ_n) be a rigidity sequence for T. If $(\varphi_{\ell_n})_*\mu \to \nu$ weakly on $\overline{G}$, then* $\mathrm{supp}(\nu) \subset \overline{\mathcal{E}}(\varphi)$.

Let us recall that all Abelian l.c.s.c. groups are metrizable. Let d be a metric.

Definition 2.1.1. We say that $g \in G$ is a *quasi-period* of a cocycle φ over T with values in G, if there exist $\delta > 0$, a rigidity sequence (ℓ_n) for T, and a sequence $0 < \varepsilon_n \to 0$, such that

$$\mu(A_n) \geq \delta, \forall n \geq 1, \text{ where } A_n = \{x \in X : d(\varphi_{\ell_n}(x), g) < \varepsilon_n\}.$$

Lemma 2.1.5. *The set of quasi-periods is included in $\mathcal{E}(\varphi)$.*

Proof. With no loss of generality we can assume that $(\varphi_{\ell_n})_*\mu \to \nu$ where ν is a probability measure on $\overline{G}$. In view of Proposition 2.1.4 it suffices to show that a quasi-period g is in the topological support of ν. Take U a neighborhood of g, and select a smaller neighborhood $g \in V \subset U$ so that $\overline{V} \subset U$. We have $\nu(U) \geq \limsup(\varphi_{\ell_n})_*(\mu)(V) = \limsup \mu(\varphi_{\ell_n}^{-1}(V)) \geq \limsup \mu(A_n) \geq \delta$. □

The following "lifting essential values" lemma can be applied when T is an irrational rotation by α, φ below is $\mathbb{R}$-valued, centered and of bounded variation (see (2.2.1)), dealing with different subsequences of the sequence (q_n) of denominators of α.

Lemma 2.1.6. *Assume that T is ergodic and let (ℓ_n) be a rigidity sequence of T. Assume that $\varphi : X \to H$ is a cocycle such that there exists a compact neighborhood $C \subset H$ of $0 \in H$ for which $\varphi_{\ell_n} \in C$ eventually. Let $\psi : X \to G$ be a cocycle such that $(\psi_{\ell_n})_*(\mu) \to \kappa$ with κ a probability measure on $\overline{G}$. Assume that*

$$0 \neq g_0 \in \mathrm{supp}(\kappa) \cap G. \tag{2.1.3}$$

Then there exists $h_0 \in H$ such that $(h_0, g_0) \in \mathcal{E}(\Phi)$, where $\Phi := (\varphi, \psi) : X \to H \times G$.

Proof. Note first that in view of Proposition 2.1.4, $g_0 \in \mathcal{E}(\psi)$. By passing to a subsequence if necessary, we can assume that the distributions of φ_{ℓ_n} and Φ_{ℓ_n} converge, that is

$$(\varphi_{\ell_n})_*(\mu) \to \nu, \quad (\Phi_{\ell_n})_*(\mu) \to \rho,$$

where ν is a probability measure on $\overline{H}$. In fact ν is concentrated on C (by our standing assumption). Hence ρ is a probability measure concentrated on $C \times \overline{G}$. Moreover,

$$\text{the projections of } \rho \text{ on } C \text{ and } \overline{G} \text{ are equal to } \nu \text{ and } \kappa \text{ respectively.} \quad (2.1.4)$$

Using (2.1.3), for each $n \geq 1$ select an open neighborhood $G \supset V_n \ni g_0$ so that $\overline{V}_n$ is compact, $\operatorname{diam} \overline{V}_n < 1/n$, $\kappa(V_n) > 0$ and $V_{n+1} \subset V_n$. In view of (2.1.4), $\rho(C \times \overline{V}_n) > 0$. Since $C \times \overline{V}_n$ is compact, there is $(c_n, g_n) \in C \times \overline{V}_n$ such that $(c_n, g_n) \in \operatorname{supp}(\rho)$ (if no such a point exists, each point of $C \times \overline{V}_n$ has a neighborhood which is of measure ρ zero, a finite union of such neighborhoods must then cover the set $C \times \overline{V}_n$, a contradiction).

In this way we obtain a sequence (c_n, g_n), $n \geq 1$, of points which are in $\operatorname{supp}(\rho) \cap C \times \overline{V}_1$ and from which we can choose a converging subsequence (c_{n_k}, g_{n_k}). Moreover, by our assumption on the diameters of V_n, $(c_{n_k}, g_{n_k}) \to (c, g_0)$, so the result follows. □

In particular, by the proof of Lemma 2.1.5, Lemma 2.1.6 will apply when $g_0 \in G$ is an essential value of ψ obtained as a quasi-period along a subsequence of the sequence (q_n) of denominators of α.

2.1.2 Essential values of cocycles taking values in $\mathbb{R}^d$

In the lemmas of this subsection, Φ will stand for a cocycle with values in $\mathbb{R}^d$.

Lemma 2.1.7. *Let $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ be a non zero essential value of Φ. Then there is a change of basis in $\mathbb{R}^d$ given by a matrix M such that the vector $(1, 0, ..., 0)$ is an essential value of the cocycle $M\Phi$. If θ is rational, then M can be taken rational.*

Proof. There is a change of basis in $\mathbb{R}^d$ with θ as the first vector of the new basis. This can be done via a matrix M_1 with rational coefficients if $\theta \in \mathbb{Z}^d$. The cocycle $\Phi' = M_1\Phi$ has an essential value of the form $(\theta_1, 0, ..., 0)$, where θ_1 is a positive real (a positive integer if θ is in $\mathbb{Z}^d$, for an adapted choice of M_1). By applying a linear isomorphism M_2 (rational in the θ rational case) we get that $\Phi'' = M_2 M_1 \Phi$ has an essential value of the form $(1, 0, ..., 0)$. □

Lemma 2.1.8. *There exist a linear isomorphism* $M : \mathbb{R}^d \to \mathbb{R}^d$ *and integers* $d_0, d_1, d_2 \geq 0$ *such that if we set* $H_i = \mathbb{R}^{d_i}$, $i = 0, 1, 2$, *then*

$$\mathbb{R}^d = H_0 \times H_1 \times H_2, \quad M\Phi = (\psi_0, \psi_1, \psi_2)$$

with $\psi_i : X \to H_i$, $i = 0, 1, 2$, *and* $\mathcal{E}(M\Phi) = \{0\} \times H_1 \times \Gamma_2$, *with* Γ_2 *a discrete subgroup of* H_2 *such that* H_2/Γ_2 *is compact. If* Φ *is a coboundary, then* $d_1 = d_2 = 0$.

Proof. The group $\mathcal{E}(\Phi)$ is a closed subgroup of $\mathbb{R}^d$, hence there are linearly independent vectors $v_1, \ldots, v_{d_1}$, $w_1, \ldots, w_{d_2}$ in $\mathbb{R}^d$ such that

$$\mathcal{E}(\Phi) = \{s_1 v_1 + \ldots + s_{d_1} v_{d_1} + t_1 w_1 + \ldots + t_{d_2} w_{d_2} : \ s_j \in \mathbb{R}, \ t_k \in \mathbb{Z}\}.$$

Select $y_1, \ldots, y_{d_0} \in \mathbb{R}^d$ so that together with previously chosen v_j and w_k we obtain a basis of $\mathbb{R}^d$. Then define a linear isomorphism M of $\mathbb{R}^d$ by setting

$$M(y_i) = e_i, \ M(v_j) = e_{d_0+j}, \ M(w_k) = e_{d_0+d_1+k},$$

where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. Since $\mathcal{E}(M\Phi) = M\mathcal{E}(\Phi)$, we obtain $\mathcal{E}(M\Phi) = \{0\} \times H_1 \times \Gamma_2$ as required and $M\Phi = (\psi_0, \psi_1, \psi_2)$. □

Corollary 2.1.9. *Let us consider the case* $d = 2$. *Let* $\Phi = (\varphi^1, \varphi^2) : X \to \mathbb{R}^2$ *be a cocycle such that* $\mathcal{E}(\Phi) \neq \{0\}$. *Then*

$$\Phi \text{ is regular if and only if } a\varphi^1 + b\varphi^2 : X \to \mathbb{R} \text{ is regular for each } a, b \in \mathbb{R}. \tag{2.1.5}$$

Proof. In view of Lemma 2.1.3 we only need to prove sufficiency. Suppose Φ is not regular. In view of Lemma 2.1.8 we obtain a linear isomorphism $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $M\Phi = (\psi^0, \psi^i)$ with $\psi^0 : X \to H_0$, $\psi^i : X \to H_i$, i equals either 1 or 2 and $H_0 \neq \{0\}$ by non-regularity of Φ and $H_i \neq \{0\}$ since $\mathcal{E}(\Phi) \neq \{0\}$ by hypothesis. Hence $\mathcal{E}(\psi^0) = \{0\}$ and there are a and b such that $\psi^0 = a\varphi^1 + b\varphi^2$. But $a\varphi^1 + b\varphi^2$ is, by assumption, regular, so ψ^0 must be a coboundary. Hence (ψ^0, ψ^i) is cohomologous to $(0, \psi^i)$ and it now follows from Remark 2.1.1 that (ψ^0, ψ^i) is regular, a contradiction. □

Lemma 2.1.10. *Let* $\Phi : X \to \mathbb{R}^d$ *be a recurrent cocycle and let* $M : \mathbb{R}^d \to \mathbb{R}^d$ *be a linear isomorphism of* $\mathbb{R}^d$ *yielding the assertions of the previous lemma. Assume additionally that the quotient cocycle* $\Phi/\mathcal{E}(\Phi)$ *is constant. Then* $\psi_0 = 0$. *Moreover,* Φ *is regular.*

Proof. Since $\mathcal{E}(M\Phi) = M\mathcal{E}(\Phi) = \{0\} \times H_1 \times \Gamma_2$, we have

$$(\psi_0(x), \psi_1(x), \psi_2(x))/\{0\} \times H_1 \times \Gamma_2 = const.$$

It follows that there is a constant $b \in \mathbb{R}^{d_0}$ such that $\psi_0 = b$. However, $M\Phi$ is recurrent as Φ is recurrent and therefore ψ_0 is also recurrent. It follows that $b = 0$. Now regularity follows from Remark 2.1.1 since $H_1 \times \Gamma_2$ has a compact quotient in $H_1 \times H_2$. □

An example of a situation described by the previous lemma is the following: let ψ be an ergodic step cocycle with values in $\mathbb{Z}$ over an irrational rotation by $\alpha \in (0, 1)$. If we modify ψ by $\mathbb{1}_{[0,\alpha)} - \alpha$ which is a coboundary, then for $\varphi := \psi + \mathbb{1}_{[0,\alpha)} - \alpha$ we have $\mathcal{E}(\varphi) = \mathcal{E}(\psi) = \mathbb{Z}$; here $\Gamma_2 = \mathbb{Z}$ and $\varphi \bmod \mathcal{E}(\varphi) = -\alpha$.

2.2 More about irrational rotation

Lemma 2.2.1. *1) Let p, q be two coprime positive integers and $\theta = q(\alpha - \frac{p}{q})$ with $|\theta| < \frac{1}{q}$. When $\theta > 0$, each interval $[\frac{j}{q}, \frac{j+1}{q})$, $0 \le j \le q-1$ contains one (and only one) number of the form $\{k\alpha\}$, with $0 \le k \le q-1$. When $\theta < 0$ the same is true for $j = 1, \dots, q-2$; there are two points $k\alpha$ (one for $k = 0$) in $[0, \frac{1}{q})$ and no such a point in $[\frac{q-1}{q}, 1)$.*

2) For each $x \in \mathbb{T}$ the distance between two consecutive elements of the set $\Big\{\{x + k\alpha\} : k = 0, \dots, q-1\Big\}$ is $< \frac{2}{q}$.

3) There are at most two elements of the set $\Big\{\{x+k\alpha\} : k = 0, \dots, q-1\Big\}$ in any interval on the circle of length $\frac{1}{q}$ (hence at most four such elements in any interval of length $\frac{2}{q}$).

4) If additionally $q = q_n$, the distance between two consecutive elements of the set $\Big\{\{x - k\alpha\} : k = 0, \dots, q\Big\}$ is $> \frac{1}{2q_n}$.

Proof. The map $k \to j(k) := kp \bmod q$ is a permutation of $\{0, 1, ..., q-1\}$. If $\theta > 0$, then $\{k\alpha\} = \{k(\frac{p}{q} + \frac{\theta}{q})\} = \frac{j(k)}{q} + \frac{k\theta}{q}$ is at distance $\frac{k\theta}{q} < \frac{1}{q}$ from $\frac{j(k)}{q}$, hence it is in the interval $[\frac{j(k)}{q}, \frac{j(k)+1}{q})$. The proof is similar if $\theta < 0$. Hence the first assertion follows.

Assertion 2) is true for $x = 0$ by 1); hence, because the distance is invariant by translations, it is true for any $x \in \mathbb{T}$.

For 3), suppose that there are $\{x+k_1\alpha\} < \{x+k_2\alpha\} < \{x+k_3\alpha\}$ distinct in an interval of length $< 1/q$. We have $\frac{\ell}{q} \le \{k_1\alpha\} < \{k_2\alpha\} < \{k_3\alpha\} < \frac{\ell+2}{q}$, for some ℓ. Either $[\frac{\ell}{q}, \frac{\ell+1}{q})$ or $[\frac{\ell+1}{q}, \frac{\ell+2}{q})$ contains two points of the set $\Big\{\{k\alpha\} : 0 \le k < q-1\Big\}$, which clearly contradicts 1).

4) We have the following

$$\frac{1}{2q_n} \le \frac{1}{q_n + q_{n-1}} \le \|q_{n-1}\alpha\| \le \|j\alpha\|, \ \forall j, |j| < q_n$$

and the assertion follows. □

The first assertion of Lemma 2.2.1 implies easily the well-known *Denjoy-Koksma inequality*: let φ be a centered function of bounded variation $V(\varphi)$ and p/q a rational number (in lowest terms) such that $|\alpha - p/q| < 1/q^2$, then

$$\left|\sum_{\ell=0}^{q-1} \varphi(x + \ell\alpha)\right| \le V(\varphi). \tag{2.2.1}$$

Indeed, let us consider the case $\theta > 0$ (the proof is analogous when $\theta < 0$). We can assume $x = 0$. For $j = 0, ..., q-1$, there is one and only one point $\{k_j\alpha\}$ of the set $\{\{k\alpha\}, k = 0, ..., q-1\}$ in $I_j := [\frac{j}{q}, \frac{j+1}{q})$. Given an interval $I \subset [0, 1)$ denote by $V(\varphi, I)$ the variation of φ on I. Notice that if $x, y \in I$ then $|\varphi(x) - \varphi(y)| \leqslant V(\varphi, I)$. Since $\int \varphi \, dt = 0$, we have:

$$\begin{aligned}
\left|\sum_{j=0}^{q-1} \varphi(j\alpha)\right| &= \left|\sum_{j=0}^{q-1} \varphi(\{k_j\alpha\}) - q \int \varphi(t) \, dt\right| = \left|\sum_{j=0}^{q-1} q \int_{j/q}^{(j+1)/q} [\varphi(\{k_j\alpha\}) - \varphi(t)] dt\right| \\
&\le q \sum_{j=0}^{q-1} \int_{j/q}^{(j+1)/q} |\varphi(\{k_j\alpha\}) - \varphi(t)| \, dt \le q \sum_{j=0}^{q-1} \int_{j/q}^{(j+1)/q} V(\varphi, I_j) \\
&= \sum_{j=0}^{q-1} V(\varphi, I_j) \leqslant V(\varphi).
\end{aligned}$$

Notation: For $\beta \in [0, 1)$, $\mathcal{L}(\beta)$ denotes the set of limit points of the sequence $(\|q_n\beta\|)_{n\ge1}$.

Another important quantity is $\beta^{(n)} := \inf_{0\le|j|<q_n} \|\beta - j\alpha\|$. We have the following properties for $\beta^{(n)}$ and the set $\mathcal{L}(\beta)$:

Lemma 2.2.2. *1) If there exists n_0 such that*

$$\beta^{(n)} = \inf_{0\le|j|<q_n} \|\beta - j\alpha\| < \frac{1}{2}\|q_{n+1}\alpha\|, \ \forall n \ge n_0, \tag{2.2.2}$$

then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$.

2) Suppose α is of bounded type.

a) If $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$, then there exist $c > 0$ and an increasing sequence (n_k) such that, for every $k \ge 1$, $\|\beta - j\alpha\| \ge c/q_{n_k}$, for $0 \le |j| \le q_{n_k}$.

b) If $\beta = \frac{t}{r}\alpha + \frac{u}{s} \in (\mathbb{Q}\alpha + \mathbb{Q}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$, then there exists $c > 0$ such that $\|\beta - j\alpha\| \ge c/q_n$, for $0 \le |j| \le q_n$ $(n \ge 1)$.

Proof. 1) For each $n \geq 1$, consider the family of intervals $I_n^j = [\{j\alpha\} - \frac{1}{2}\|q_{n+1}\alpha\|, \{j\alpha\} + \frac{1}{2}\|q_{n+1}\alpha\|]$, $j = -q_n + 1, ..., q_n - 1$.

Let $n \geq n_0$. If $j \in \{-q_n + 1, ..., q_n - 1\}$ and $j' \in \{-q_{n+1} + 1, ..., q_{n+1} - 1\}$ are distinct, then the intervals I_n^j and $I_{n+1}^{j'}$ are disjoint, since otherwise by $\|(j'-j)\alpha\| \leq \frac{1}{2}\|q_{n+1}\alpha\| + \frac{1}{2}\|q_{n+2}\alpha\| < \|q_{n+1}\alpha\|$, with $0 < |j'-j| < q_n + q_{n+1} \leq q_{n+2}$ which contradicts (1.9.4).

By (2.2.2), there is a sequence $(j_n)_{n \geq n_0}$, such that $0 \leq |j_n| < q_n$ and $\beta \in I_n^{j_n}$ for $n \geq n_0$.

Since $\beta \in I_n^{j_n} \cap I_{n+1}^{j_{n+1}}$, we have $j_0 := j_{n_0} = j_{n_1} = ...$ This implies $\beta = \{j_0\alpha\}$ which completes the proof of 1).

2a) By part 1) if $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$, it follows that there exists a sequence (n_k) such that $\|\beta - j\alpha\| \geq \frac{1}{2}\|q_{n_k+1}\alpha\|$, for $0 \leq |j| \leq q_{n_k}$ and $k \geq 1$. Suppose additionally that α is of bounded type. Since $\|q_{n_k+1}\alpha\|$ and $\frac{1}{q_{n_k}}$ are comparable, there is $c > 0$ such that $\|\beta - j\alpha\| \geq c/q_{n_k}$ for $0 \leq |j| \leq q_{n_k}$.

2b) Now let $\beta = \frac{t}{r}\alpha + \frac{u}{s} \notin \mathbb{Z}\alpha + \mathbb{Z}$ with t, r, u, s integers and $r, s \geq 1$. Let j_n be such that $\varepsilon_n := \min_{j: 0 \leq |j| \leq q_n} \|\frac{t}{r}\alpha + \frac{u}{s} - j\alpha\| = \|\frac{t}{r}\alpha + \frac{u}{s} - j_n\alpha\| > 0$.

We have $\frac{t}{r}\alpha + \frac{u}{s} = j_n\alpha + \ell_n \pm \varepsilon_n$, for an integer ℓ_n; hence: $(rsj_n - ts)\alpha = ru - rs\ell_n \pm rs\varepsilon_n$. It follows

$$\|(rsj_n - ts)\alpha\| \leq rs\,|\varepsilon_n|. \tag{2.2.3}$$

Suppose that $rsj_n - ts = 0$ for infinitely many n. Then $\frac{t}{r} = j_n$ and $|u - s\ell_n| = s|\varepsilon_n|$. Since $|\varepsilon_n|$ is arbitrarily small for n large enough and u, s, ℓ_n are integers, it follows $u = s\ell_n$. Then, we find $\beta = j_n\alpha + \ell_n$ contrary to the assumption that β is not in $\mathbb{Z}\alpha + \mathbb{Z}$. It follows that the integers $rsj_n - ts$ are different from zero for all $n \geq n_1$.

Now, α is of bounded type, so there is $K > 0$ such that $q_{n+rs+1} \leq K\,q_n$, for every $n \geq 1$. Using additionally (1.9.3) and (1.9.4), we obtain

$$\frac{1}{2K\,q_n} \leq \frac{1}{2q_{n+rs+1}} \leq \|q_{n+rs}\,\alpha\| \leq \|k\alpha\|, \text{ for } 1 \leq |k| < q_{n+rs+1}. \tag{2.2.4}$$

On the other hand, in view of (1.9.1), given any constant $C > 0$ we have

$$q_{n+m} \geq mq_n + C \tag{2.2.5}$$

for all $m \geq 1$ and n large enough (indeed, it suffices to consider n so that $q_{n-1} \geq C$). Hence, for the integer $|rsj_n - ts|$ we have

$$0 < |rsj_n - ts| \leq rsq_n + |t|s \leq q_{n+rs+1}$$

whenever n is large enough. Therefore, for n large enough, by (2.2.3) and (2.2.4), we obtain

$$|\varepsilon_n| \geq c/q_n, \text{ with } c = \frac{1}{2K}.$$

By taking $c > 0$ smaller if necessary, the conclusion holds for all $n \geq 1$. □

Lemma 2.2.3. *Suppose α is of bounded type. Let B be a non-empty finite subset of $(\mathbb{Q}\beta + \mathbb{Q}\alpha + \mathbb{Q}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$, where β is a real number. Then there exist $c > 0$ and a strictly increasing sequence (n_k) such that*

$$\forall \beta_i \in B, \ \forall k \geq 1, \ \|\beta_i - j\alpha\| \geq c/q_{n_k}, \ \textit{for } 0 \leq |j| \leq q_{n_k}.$$

Proof. We have $B = B_0 \cup B_1$, where

$$B_0 = \{\beta_i; \beta_i = \frac{u_i}{s_i}\alpha + \frac{w_i}{s_i} \text{ with } u_i, w_i, s_i \in \mathbb{Z}, s_i \neq 0, \beta_i \notin \mathbb{Z}\alpha + \mathbb{Z}\}$$
$$B_1 = \{\beta_i; \beta_i = \frac{v_i}{s_i}\beta + \frac{u_i}{s_i}\alpha + \frac{w_i}{s_i}, \text{ with } v_i, u_i, w_i, s_i \in \mathbb{Z} \text{ and } v_i, s_i \neq 0\}.$$

Remark that B_0 or B_1 can be empty and that $B = B_0$ if $\beta \in \mathbb{Q}\alpha + \mathbb{Q}$.

If $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ and B_1 is not empty, we apply Lemma 2.2.2 to $\beta' := (\prod v_\ell)\beta$. There are a positive constant c and a sequence (n_k) such that

$$\|\beta' - j\alpha\| \geq \frac{c}{q_{n_k}}, \ 0 \leq |j| < q_{n_k}.$$

Let $M = (\max s_\ell)(\prod v_\ell)$, $M_i = s_i \prod_{\ell \neq i} v_\ell$. We have $\frac{v_i}{s_i}\beta = \frac{\beta'}{M_i}$.

Since $L_i := M_i \frac{u_i}{s_i}$ and $M_i \frac{w_i}{s_i}$ are integers, we have for j such that $0 \leq |M_i j - L_i| < q_{n_k}$:

$$\begin{aligned} M_i \left\| \frac{v_i}{s_i}\beta + \frac{u_i}{s_i}\alpha + \frac{w_i}{s_i} - j\alpha \right\| &\geq \left\| M_i \frac{v_i}{s_i}\beta - M_i \left(j\alpha - \frac{u_i}{s_i}\alpha - \frac{w_i}{s_i} \right) \right\| \\ &= \|\beta' - (M_i j - L_i)\alpha\| \geq \frac{c}{q_{n_k}}. \end{aligned}$$

We have $M_i|j| + |L_i| \leq M|j| + L$, with $L := \max |L_i|$. As α is of bounded type, there are r and K such that $Mq_{n-r} + L \leq q_n \leq Kq_{n-r}$, for all $n \geq 1$. This implies, simultaneously for every i:

$$\left\| \frac{v_i}{s_i}\beta + \frac{u_i}{s_i}\alpha + \frac{w_i}{s_i} - j\alpha \right\| \geq \frac{1}{M_i} \|\beta' - (M_i j - L_i)\alpha\| \geq \frac{c}{MK} \frac{1}{q_{n_k - r}}, \text{ if } |j| < q_{n_k - r}.$$

For β_i in B_0, if this subset is non empty, by the part 2b) of the previous lemma any subsequence of (q_n) is "good".

We conclude that the subsequence $(q_{n_k - r})_{r \geq 1}$ fulfills the assertion of the lemma. □

Remark 2.2.1. As the proof of Lemma 2.2.3 shows, the result is true for any change of the part belonging to $\mathbb{Q}\alpha+\mathbb{Q}$ for the elements of B_1 (that is, we may replace $\frac{u_i}{s_i}\alpha + \frac{w_i}{s_i}$, for $i = 1, ..., t$, by a different element of $\mathbb{Q}\alpha + \mathbb{Q}$). However, each time we change this part, we also change the resulting subsequence (q_{n_k}).

Remark 2.2.2. When α is not of bounded type, the set $K(\alpha) = \{\beta \in \mathbb{R} : \lim_n \|q_n\beta\| = 0\}$ is an uncountable additive subgroup of $\mathbb{R}$.

Nevertheless, if $\lim_n \|q_n\beta\| = 0$ and $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$, the rate of convergence toward 0 is moderate, as shown by the following lemma (see [6], [22], [21], [7]).

Lemma 2.2.4. *If there exists n_0 such that $\|q_n\beta\| \leq \frac{1}{4}q_n\|q_n\alpha\|$ for $n \geq n_0$, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$. In particular, if α is of bounded type and β satisfies $\lim_n \|q_n\beta\| = 0$, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$.*

2.3 Step cocycles over an irrational rotation

In this section, we study the regularity of a step $\mathbb{R}^d$-valued cocycle $\Phi = (\varphi^1, \dots, \varphi^d)$ over an irrational rotation $T : x \to x + \alpha$. For such a cocycle the coordinate $\mathbb{R}$-valued cocycles φ^j are integrable and we will constantly assume that $\int_0^1 \varphi^j \, d\mu = 0$ with $\mu = m_{\mathbb{T}}$ the Lebesgue measure on $\mathbb{T}^1$, for $j = 1, \dots, d$.

2.3.1 Representations of step cocycles

The coordinates of $\Phi = (\varphi^1, \dots, \varphi^d)$ can be (uniquely) represented as follows:

$$\varphi^j(x) = \sum_i t_{i,j} \, (1_{I_{i,j}}(x) - \mu(I_{i,j})), \tag{2.3.1}$$

where, for $j = 1, \dots, d$, $\{I_{i,j}\}$ is a finite family of disjoint intervals of $[0, 1)$ (covering $[0, 1)$ and maximal on which φ^j is constant) and $t_{i,j} \in \mathbb{R}$. Clearly, when $d \geq 1$ is fixed, the family of step cocycles form a linear space over $\mathbb{R}$.

Setting $\beta_{i,j} = \mu(I_{i,j})$ and $\psi^{i,j} = 1_{I_{i,j}} - \beta_{i,j}$, we have $\psi_n^{i,j}(x) = \sum_{k=0}^{n-1} 1_{I_{i,j}}(x+k\alpha) - n\beta_{i,j}$; hence the cocycle φ_n^j can be written in the following form:

$$\varphi_n^j(x) = \sum_i t_{i,j} \, \psi_n^{i,j}(x) = \sum_i t_{i,j} \, (u_{(n)}^{i,j}(x) - \{n\beta_{i,j}\}), \tag{2.3.2}$$

with the notation (which is not a cocycle expression)

$$u_{(n)}^{i,j}(x) := \psi_n^{i,j}(x) + \{n\beta_{i,j}\} = \sum_{k=0}^{n-1} 1_{I_{i,j}}(x + k\alpha) - [n\beta_{i,j}] \in \mathbb{Z}. \tag{2.3.3}$$

Remark 2.3.1. *Without loss of generality, we can assume that the difference between any two discontinuity points of the cocycle Φ is never a multiple of α (modulo 1).* Indeed, if β and β' are two discontinuity points of a component of Φ such that $\beta' - \beta \in \mathbb{Z}\alpha + \mathbb{Z}$, we can suppress one of them by adding to Φ a coboundary, without changing the ergodic properties of Φ (we use the fact that $\mathbb{1}_{[\beta,\beta')}(x) - (\beta' - \beta)$ is coboundary[31]). In particular, after modification, the lengths $\mu(I_{i,j})$ in the representation of the new cocycle are not in $\mathbb{Z}\alpha + \mathbb{Z}$.

2.3.2 Rational step cocycles

Assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is a zero mean step cocycle with its unique representation (2.3.1) of the form

$$\varphi = \sum_{i=1}^{m} t_i(\mathbb{1}_{I_i} - \mu(I_i)). \tag{2.3.4}$$

Definition 2.3.1. We say that φ is *rational* if there are $c_i \in \mathbb{Q}$, $i = 1, ..., m$ and $\beta \in \mathbb{R}$ such that

$$\varphi = \sum_{i=1}^{m} c_i \mathbb{1}_{I_i} - \beta. \tag{2.3.5}$$

Lemma 2.3.1. *Assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is a (zero-mean) step cocycle. The following conditions are equivalent:*

(i) φ is rational.

(ii) There exists $w \in \mathbb{R}$ such that in the unique representation (2.3.4) of φ we have $t_i \in w + \mathbb{Q}$ for $i = 1, ..., m$.

(iii) φ takes values in a coset of $\mathbb{Q}$.

In particular, the family of rational cocycles is a linear space over $\mathbb{Q}$.

Proof. (i)⇒(ii) By (2.3.4), $\varphi = \sum_{i=1}^{m} t_i \mathbb{1}_{I_i} - \gamma$, where $\gamma = \sum_{i=1}^{m} t_i \mu(I_i)$. For $x \in I_i$ we have

$$c_i - \beta = \varphi(x) = t_i - \gamma,$$

so $t_i \in (\gamma - \beta) + \mathbb{Q}$ for $i = 1, ..., m$.

[31] Indeed, we have $\mathbb{1}_{[1-\alpha,1)}(x) - \alpha = j(x) - j(x + \alpha)$ with $j(x) = \{x\}$, then for integers k, s

$$\begin{aligned}\mathbb{1}_{[1-\{k\alpha+s\},1)}(x) - \{k\alpha + s\} &= \mathbb{1}_{[1-\{k\alpha\},1)}(x) - \{k\alpha\} \\ &= j(x) - j(x + k\alpha) = j_k(x) - j_k(x + \alpha).\end{aligned}$$

The general case is obtained using the obvious fact that other rotations commute with $Tx = x + \alpha$.

(ii)⇒(iii) For some $r_i \in \mathbb{Q}$, $i = 1, ..., m$ and $x \in [0, 1)$ we have

$$\varphi(x) = \sum_{i=1}^{m} (w + r_i)(\mathbb{1}_{I_i}(x) - \mu(I_i)) = \sum_{i=1}^{m} r_i \mathbb{1}_{I_i}(x) + (w - \gamma) \in (w - \gamma) + \mathbb{Q}.$$

(iii)⇒(i) Take the unique representation (2.3.4) of φ: $\varphi = \sum_{i=1}^{m} t_i \mathbb{1}_{I_i} - \gamma$ with $\gamma = \sum_{i=1}^{m} t_i \mu(I_i)$. By assumption, there exists $\eta \in \mathbb{R}$ such that $\varphi(x) \in \eta + \mathbb{Q}$ for each $x \in [0, 1)$. Thus, for $x \in I_i$ we have

$$t_i - \gamma = \varphi(x) = \eta + r_i$$

for some $r_i \in \mathbb{Q}$. Whence $t_i \in (\gamma + \eta) + \mathbb{Q}$ for $i = 1, ..., m$.

The latter assertion follows directly from (iii). □

Suppose that φ is rational with a representation (2.3.5) and let $\varphi = \sum_{i=1}^{m} c_i' \mathbb{1}_{I_i} - \beta'$ (with $c_i' \in \mathbb{Q}$) be another rational representation. Then by (iii) of Lemma 2.3.1 it follows that $\beta - \beta' \in \mathbb{Q}$, in other words, in the rational representation (2.3.5) the coset $\beta + \mathbb{Q} \in \mathbb{R}/\mathbb{Q}$ is unique. By $\beta(\varphi)$ we will denote that coset (in fact, less formally it will be the number β in (2.3.5) understood modulo $\mathbb{Q}$). Note that

$$\varphi(x) \in \beta(\varphi) \text{ for all } x \in \mathbb{T}.$$

With this in mind we have immediately the following observation:

Lemma 2.3.2. *Assume that $\varphi^1, ..., \varphi^d : \mathbb{T} \to \mathbb{R}$ are rational step cocycles. Assume moreover that $a_j \in \mathbb{Q}$ for $j = 1, ..., d$ and set $\varphi = \sum_{j=1}^{d} a_j \varphi^j$. Then*

$$\beta(\varphi) = \sum_{j=1}^{d} a_j \beta(\varphi^j).$$

Now, let $d \geq 1$.

Definition 2.3.2. We say that a step cocycle $\Phi : \mathbb{T} \to \mathbb{R}^d$ is a *rational step cocycle* if its coordinates φ^j are rational, i.e.:

$$\varphi^j = \sum_i c_{i,j} \mathbb{1}_{I_{i,j}} - \beta_j, \tag{2.3.6}$$

where the coefficients $c_{i,j}$ are rational numbers and β_j is such that $\int_0^1 \varphi^j d\mu = 0$, $j = 1, ..., d$.

In this case, by replacing Φ by its non-zero integer multiple so that all $c_{i,j}$ are integers (recall that a non-zero multiple of a cocycle Φ shares its ergodic properties with Φ) we obtain:

$$\varphi_n^j(x) = u_{(n)}^j(x) - \{n\beta_j\},\ n \geq 1, \tag{2.3.7}$$

where the functions $u_{(n)}^j$ have values in $\mathbb{Z}$.

Below we write $\beta_j = \beta_j(\varphi^j) = \beta_j(\Phi)$ (in the representation (2.3.6)) to stress the dependence of the β_j's on the cocycle Φ. The number of discontinuities of Φ is denoted $D(\Phi)$.

We denote by $\mathcal{L}(\beta_j)$ the set of limit values of the sequence $(\|q_n\beta_j\|)_{n\geq 1}$. Observe that if $\mathcal{L}(\beta_j) \neq \{0\}$, there exists a sequence (n_k) such that $\lim_k\{q_{n_k}\beta_j\} \in (0,1)$. Let $L := \max_{i,j} V(\psi^{i,j})$ in case (2.3.2), or $L := \max_j V(\varphi^j)$ in case (2.3.7), where V is the variation.

$\mathcal{F}$ will denote the interval of integers

$$\mathcal{F} = \{\ell \in \mathbb{Z} : |\ell| \leq L+1\}. \tag{2.3.8}$$

From (2.2.1), (2.3.7) and (2.3.3), it follows that:

$$u_{(q_n)}^j(x) \in \mathcal{F},\ u_{(q_n)}^{i,j}(x) \in \mathcal{F}. \tag{2.3.9}$$

Lemma 2.3.3. *Let Φ be a rational step cocycle. If $\mathcal{L}(\beta_{j_0}) \neq \{0\}$ for some j_0, then $\mathcal{E}(\Phi)$ contains a rational vector $\theta = (\theta_1, ..., \theta_d)$ with $\theta_{j_0} \neq 0$.*

Proof. By multiplying Φ by an integer if needed, we can use (2.3.7) with $u_{(n)}^j(x) \in \mathbb{Z}$. We can select a subsequence (n_k) so that $(\{q_{n_k}\beta_j\})_{k\geq 1}$ converges for all $j = 1, \dots, d$ to a limit denoted δ_j, with $\delta_{j_0} \in (0,1)$. Taking into account (2.3.9), denote for $(\ell_1, \dots, \ell_d) \in \mathcal{F}^d$

$$A_{k,\ell_1,\dots,\ell_d} = \{x \in \mathbb{T} : u_{(q_{n_k})}^j(x) = \ell_j,\ j = 1, \dots, d\}.$$

Note that, for each $k \geq 1$, $\{A_{k,\ell_1,\dots,\ell_d} : (\ell_1, \dots, \ell_d) \in \mathcal{F}^d\}$ is a partition of $\mathbb{T}$. By passing to a further subsequence if necessary, we can assume that $\mu(A_{k,\ell_1,\dots,\ell_d}) \to \gamma_{\ell_1,\dots,\ell_d}$ when $k \to \infty$, for each $(\ell_1, \dots, \ell_d) \in \mathcal{F}^d$. In view of (2.3.8), (2.3.7) and the fact that $\int \varphi^j\, d\mu = 0$, we have

$$\begin{aligned}&\sum_{\ell\in\mathcal{F}} \ell\, \mu\left(\cup_{\ell_1,\dots,\ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d\in\mathcal{F}} A_{k,\ell_1,\dots,\ell_{j_0-1},\ell,\ell_{j_0+1},\dots,\ell_d}\right)\\ &= \int_0^1 u_{(q_{n_k})}^{j_0}(x)\, dx = \int_0^1 \left(\varphi_{q_{n_k}}^{j_0}(x) + \{q_{n_k}\beta_{j_0}\}'\right) dx = \{q_{n_k}\beta_{j_0}\}' \to \delta_{j_0}.\end{aligned}$$

It follows that

$$\sum_{\ell\in\mathcal{F}} \ell \sum_{\ell_1,\dots,\ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d\in\mathcal{F}} \gamma_{\ell_1,\dots,\ell_{j_0-1},\ell,\ell_{j_0+1},\dots,\ell_d} = \delta_{j_0} \tag{2.3.10}$$

with $\delta_{j_0} \in (0,1)$. Hence there are $\underline{\ell} \neq \underline{\ell}'$ such that

$$\sum_{\ell_1,\dots,\ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d \in \mathcal{F}} \gamma_{\ell_1,\dots,\ell_{j_0-1},\underline{\ell},\ell_{j_0+1},\dots,\ell_d} > 0$$
$$\sum_{\ell_1,\dots,\ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d \in \mathcal{F}} \gamma_{\ell_1,\dots,\ell_{j_0-1},\underline{\ell}',\ell_{j_0+1},\dots,\ell_d} > 0;$$

Indeed otherwise, $\sum_{\ell_1,\dots,\ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d \in \mathcal{F}} \gamma_{\ell_1,\dots,\ell_{j_0-1},\ell_0,\ell_{j_0+1},\dots,\ell_d} = 1$ for some $\ell_0 \in \mathcal{F}$ and the other sums are 0, so that the left hand side of (2.3.10) is an integer, a contradiction. This implies

$$\gamma_{\ell_1,\dots,\ell_{j_0-1},\underline{\ell},\ell_{j_0+1},\dots,\ell_d} > 0, \ \gamma_{\ell'_1,\dots,\ell'_{j_0-1},\underline{\ell}',\ell'_{j_0+1},\dots,\ell'_d} > 0,$$

for some $d-1$-uples $(\ell_1,\dots \ell_{j_0-1},\ell_{j_0+1},\dots,\ell_d)$ and $(\ell'_1,\dots \ell'_{j_0-1},\ell'_{j_0+1},\dots,\ell'_d)$. By Lemma 2.1.5 it follows that

$$(\ell_1-\delta_1,\dots,\ell_{j_0-1}-\delta_{j_0-1},\underline{\ell}-\delta_{j_0},\ell_{j_0+1}-\delta_{j_0+1},\dots,\ell_d-\delta_d) \in \mathcal{E}(\Phi),$$
$$(\ell'_1-\delta_1,\dots,\ell'_{j_0-1}-\delta_{j_0-1},\underline{\ell}'-\delta_{j_0},\ell'_{j_0+1}-\delta_{j_0+1},\dots,\ell'_d-\delta_d) \in \mathcal{E}(\Phi).$$

Thus $(\ell_1-\ell'_1,\dots,\underline{\ell}-\underline{\ell}',\dots,\ell_d-l'_d) \in \mathcal{E}(\Phi)$ with $\underline{\ell}-\underline{\ell}' \neq 0$ which completes the proof (for the initial Φ we have to divide by an integer and obtain a non zero essential value with rational coordinates). □

Theorem 2.3.4. *Let Φ be a rational step cocycle with values in $\mathbb{R}^d$. There are $d(\Phi)$, $0 \leq d(\Phi) \leq d$, and a change of basis of $\mathbb{R}^d$ given by a rational matrix M such that $M\Phi = (\hat{\varphi}^1,...,\hat{\varphi}^{d(\Phi)},\hat{\varphi}^{d(\Phi)+1},...,\hat{\varphi}^d)$ satisfies:*

1) $\mathcal{E}(M\Phi)$ contains the subgroup generated by

$$(1,0,...,0),(0,1,0,...,0),...,(\underbrace{0,0,...,1}_{d(\Phi)},0,...,0),$$

2) the cocycle $\hat{\Phi} = (\hat{\varphi}^{d(\Phi)+1},...,\hat{\varphi}^d)$ is a rational cocycle like (2.3.6) and satisfies $\lim_n \|q_n\beta_j(\hat{\Phi})\| = 0$ for $j = d(\Phi)+1,\dots,d$.

Proof. We will apply successively Lemmas 2.3.3, 2.1.7 and 2.3.2. If $\mathcal{L}(\beta_j) = \{0\}$ for all $j = 1,\dots,d$, we put $d(\Phi) = 0$. Suppose not all $\mathcal{L}(\beta_j)$ are equal to $\{0\}$, say $\mathcal{L}(\beta_1) \neq \{0\}$. Then by Lemma 2.3.3, there is a rational vector $\theta = (\theta_1,\dots,\theta_d) \in \mathcal{E}(\Phi)$ with $\theta_1 \neq 0$.

Take a linear (rational) isomorphism M_1 of $\mathbb{R}^d$, so that $M_1(\theta) = e_1$, where $e_1 = (1,0,...,0)$ and consider $M_1(\Phi) = (\varphi'_1,\dots,\varphi'_d)$. The step cocycles $\varphi'_2,\dots,\varphi'_d$ have their own representation (2.3.6) with β'_j instead of β_j. We now look at $\mathcal{L}(\beta'_j)$ for $j = 2,\dots,d$. If all these sets are equal to $\{0\}$, we set

$d(\Phi) = 1$ and the proof is finished. Suppose not all $\mathcal{L}(\beta'_j)$ for $j = 2, \ldots, d$ are equal to zero, say $\mathcal{L}(\beta'_2) \neq 0$. We apply Lemma 2.3.3 to $M_1(\Phi)$ and obtain $\theta' = (\theta'_1, \theta'_2, \ldots) \in \mathcal{E}(M_1\Phi)$ with $\theta'_2 \neq 0$. Note that e_1 and θ' are linearly independent. Then consider a linear (rational) isomorphism M_2 of $\mathbb{R}^d$ that fixes e_1 and sends θ' to e_2 and set

$$M_2(M_1(\Phi)) = (\varphi'_1, \varphi''_2, \ldots, \varphi''_d)$$

(this cocycle has e_1 and e_2 as its essential values).

Again, these new cocycles (except for φ'_1) have their own representation (2.3.6) with β''_j for $j = 2, \ldots, d$. We now look at $\mathcal{L}(\beta''_j)$ for $j = 3, \ldots, d$. If the sets $\mathcal{L}(\beta''_j)$, $j = 3, \ldots, d$, are equal to $\{0\}$ we set $d_1 = 2$ and the proof is complete. If not, say $\mathcal{L}(\beta''_3) \neq \{0\}$, we obtain a rational vector $\theta'' = (\theta''_1, \theta''_2, \theta''_3, \ldots) \in \mathcal{E}(M_2(M_1(\Phi)))$ with $\theta''_3 \neq 0$. Then consider a (rational) linear isomorphism $M_3 : \mathbb{R}^d \to \mathbb{R}^d$ fixing e_1, e_2 and sending θ''_3 into e_3 and pass to the cocycle $M_3(M_2(M_1(\Phi)))$. We complete the proof in finitely many steps. □

Reduction in the bounded type case

If we find $d(\Phi) = d$ in Theorem 2.3.4, then the group $\mathcal{E}(\Phi)$ contains a subgroup with compact quotient in $\mathbb{R}^d$ and hence the cocycle Φ is regular. This is the situation of the following theorem:

Theorem 2.3.5. *Let α be of bounded type. Let $(\beta_1, ..., \beta_d)$ be such that there is no non trivial rational relation between $1, \alpha, \beta_1, ..., \beta_d$. Then the cocycle $\Phi = (1_{[0,\beta_j)} - \beta_j)_{j=1,...,d}$ is regular. Every step cocycle φ with discontinuities at $\{0, \beta_1, ..., \beta_d\}$ and dimension $d' \leq d$ is regular.*

Proof. We use the notation of Theorem 2.3.4. If $d(\Phi) < d$, then we have $\lim_n \|q_n\beta_j(\hat{\Phi})\| = 0$ for $j = d(\Phi) + 1, \ldots, d$. As α is of bounded type, taking into account Lemma 2.3.2 and Lemma 2.2.4, we find a non trivial rational relation between the numbers $\beta_j(\hat{\Phi})$. Since the changes of basis are given by rational matrices M, this gives a non trivial rational relation between $1, \alpha, \beta_1, ..., \beta_d$, contrary to the assumption of the theorem. Therefore $d(\Phi) = d$ and the cocycle Φ is regular. For the second statement, observe that, if $d' \leq d$, φ is the image of Φ by a linear map (c.f.(2.3.1)). It is regular if Φ is regular by Lemma 2.1.3. □

Remark 2.3.2. a) *Application to non rational cocycles*

The previous proof is based on the concept of rational cocycle, but applies even to non rational cocycle.

As an illustration of the result, let us consider the cocycle: $\varphi = \theta \mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\theta\beta)}$, with $(1, \alpha, \beta, \theta\beta)$ rationally independent, $\theta \notin \mathbb{Q}$ and $\beta, \theta\beta \in [0, 1)$. This cocycle is not rational, but obtained from the cocycle $\Phi = (\mathbb{1}_{[0,\beta)} - \beta, \mathbb{1}_{[0,\theta\beta)} - \theta\beta)$ by the map $(y_1, y_2) \to \theta y_1 - y_2$. For α of bounded type, Φ and therefore also φ are regular (Theorem 2.3.5).

b) In the bounded type case, the reduction given by Theorem 2.3.4 reduces to a cocycle of the form (2.3.6) such that $\beta_j \in \mathbb{Z}\alpha + \mathbb{Z}$ for all $j = d_1 + 1, ..., d$. We can even obtain $\beta_j = 0$ by using the identity: $\alpha = \mathbb{1}_{[1-\alpha,1)}(x) + j(x + \alpha) - j(x)$, for $0 < \alpha < 1$, where $j(x) = \{x\}$.

Reduction in the unbounded type case, $\|q_n\beta_j\| \to 0$

If $\|q_n\beta_j\| \to 0$, $\forall j$ and $\beta_j \notin \mathbb{Z}\alpha + \mathbb{Z}$ (a situation which can occur only for α not of bounded type), the previous method of reduction cannot be applied. Nevertheless, there is a first step reduction, based on another method.

Lemma 2.3.6. *Let Φ be a step function with $D = D(\Phi)$ points of discontinuity. We have $\mu(A_{q,\ell}(\Phi)) > 1 - 2Dq\varepsilon$, with $\varepsilon = \ell\|q\alpha\|$, where*

$$A_{q,\ell}(\Phi) := \bigcap_{1\le s\le \ell} \{x \in \mathbb{T} : \Phi_q(x) = \Phi_q(x + sq\alpha)\},\ \ell,\ q \ge 1.$$

Proof. Let Δ be the set of discontinuities of Φ. If $x \notin A_{q,\ell}(\Phi)$, we can find s, $1 \le s < \ell$, and j, $0 \le j < q$, such that $\Phi(x + j\alpha) \ne \Phi(x + j\alpha + sq\alpha)$. This implies that Φ has a discontinuity at δ on the circle between $x + j\alpha$ and $x + j\alpha + sq\alpha$, and therefore x belongs to the interval $(\delta - j\alpha - \varepsilon, \delta - j\alpha + \varepsilon)$ because $\sup_{1\le s\le \ell} \|sq\alpha\| \le \ell\|q\alpha\|$. Now, the complement of $A_{q,\ell}(\Phi)$ is included in the set $\bigcup_{0\le j<q,\, t\in\Delta} B(t - j\alpha, \varepsilon)$, whose measure is less than $2Dq\varepsilon$. □

Proposition 2.3.7. *Let $\Phi = (\mathbb{1}_{[0,\beta_j)} - \beta_j, j = 1, ..., d)$. Suppose $\beta_j \notin \mathbb{Z}\alpha + \mathbb{Z}$ and $\|q_n\beta_j\| \to 0$, $\forall j$, then $\mathcal{E}(\Phi)$ contains a non zero vector in $\mathbb{Z}^d$ or a non discrete subgroup of $\mathbb{R}^d$.*

Proof. For $n \ge 1$, we can write (cf. 2.3.7): $\varphi^j_n(x) = \tilde{u}^j_{(n)}(x) - \varepsilon\|n\beta_j\|$, where $\varepsilon = \pm 1$ and $\tilde{u}^j_{(n)}$ is the integer valued function

$$\tilde{u}^j_{(n)} = u^j_{(n)} \text{ if } \{n\beta_j\} = \|n\beta_j\|,\quad \tilde{u}^j_{(n)} = u^j_{(n)} + 1 \text{ if } \{n\beta_j\} = 1 - \|n\beta_j\|.$$

a) If $\mu(\{x \in \mathbb{T} : \tilde{u}^{j_0}_{(q_n)}(x) = 0\}) \not\to 1$ for some j_0, the proof is similar to the proof of Lemma 2.3.3: by passing to a subsequence if necessary to ensure the convergence of all components $\varphi^j_{q_n}$, we find that Φ has a quasi-period

$(\rho_1, \rho_2, ..., \rho_d)$, with $\rho_{j_0} \neq 0$. It follows that $\mathcal{E}(\Phi)$ contains a non-zero vector in $\mathbb{Z}^d$.

b) Now, we can assume $\lim_n \mu(\{x \in \mathbb{T} : \tilde{u}^j_{(q_n)}(x) = 0\}) = 1$, for every j.

By Lemma 2.2.4 there is a sequence (n_k) such that $\|q_{n_k}\beta_1\| > \frac{1}{4} q_{n_k} \|q_{n_k}\alpha\|$. We put $L_k = [\eta / \|q_{n_k}\beta_1\|],\ k \geq 1$, where η is such that $\eta < \frac{1}{16D}$.

There is at least one index j_0 such that, infinitely often, $\|q_{n_k}\beta_{j_0}\|$ is the biggest value of the set $\{\|q_{n_k}\beta_j\|, j = 1, ..., D(\Phi)\}$. Hence for j_0 and an infinite subsequence, still denoted (n_k), we have

$$0 < \|q_{n_k}\beta_j\| \leq \|q_{n_k}\beta_{j_0}\|, \forall j.$$

In particular, we have $\|q_{n_k}\beta_{j_0}\| \geq \frac{1}{4} q_{n_k} \|q_{n_k}\alpha\|$.

Using the notation and the assertion of Lemma 2.3.6, we have

$$\mu(A_{q_{n_k}, L_k}(\Phi)) > 1 - 2D q_{n_k} L_k \|q_{n_k}\alpha\| \geq 1 - 8D\eta \geq \frac{1}{2}.$$

Moreover, using the definition of $A_{q_{n_k}, L_k}(\Phi)$, for $x \in A_{q_{n_k}, L_k}(\Phi)$ and $\ell \leq L_k$, we have

$$\Phi_{\ell q_{n_k}}(x) = \ell \Phi_{q_{n_k}}(x) = (\ell \tilde{u}^j_{(q_{n_k})}(x) - \varepsilon \ell \|q_{n_k}\beta_j\|, j = 1, ..., d)$$

with $\varepsilon = \pm 1$. Let $\rho \in (0, \eta)$. Put $\ell_k := [\rho / \|q_{n_k}\beta_{j_0}\|] < \eta / \|q_{n_k}\beta_{j_0}\| \leq L_k + 1$. We have, for $x \in A_{q_{n_k}, L_k}(\Phi)$, outside of a set of measure tending to 0,

$$\varphi^{j_0}_{\ell_k q_{n_k}}(x) = \ell_k \varphi^{j_0}_{q_{n_k}}(x) = \ell_k \tilde{u}^{j_0}_{(q_{n_k})}(x) - \varepsilon \ell_k \|q_{n_k}\beta_{j_0}\| = \pm [\rho / \|q_{n_k}\beta_{j_0}\|]\, \|q_{n_k}\beta_{j_0}\| \to \pm \rho.$$

For the other components $j \neq j_0$, outside of a set of measure tending to 0, we have on $A_{q_{n_k}, L_k}(\Phi)$,

$$\varphi^j_{\ell_k q_{n_k}}(x) = \ell_k \varphi^j_{q_{n_k}}(x) = \ell_k \tilde{u}^j_{(q_{n_k})}(x) - \varepsilon \ell_k \|q_{n_k}\beta_j\| = \pm \|q_{n_k}\beta_j\|\, [\rho / \|q_{n_k}\beta_{j_0}\|].$$

The above quantity is bounded, since $\|q_{n_k}\beta_j\|\, [\rho / \|q_{n_k}\beta_{j_0}\|] \leq \rho\, \|q_{n_k}\beta_j\| / \|q_{n_k}\beta_{j_0}\| \leq \rho$. Passing to a subsequence still denoted (n_k) if necessary, we obtain that outside of a set of measure tending to 0, on $A_{q_{n_k}, L_k}(\Phi)$ the sequence $(\Phi_{\ell_k\, q_{n_k}}(x))$ converges to the vector $(\rho_1, \rho_2, ..., \rho_d)$.

Now, the measure of $A_{q_{n_k}, L_k}(\Phi)$ is bounded away from 0 and the sequence $(\ell_k q_{n_k})$ is a rigidity sequence for T, since $\ell_k \leq L_k + 1$ and

$$L_k \|q_{n_k}\alpha\| \leq \eta \frac{\|q_{n_k}\alpha\|}{\|q_{n_k}\beta_1\|} \leq 4 \frac{\eta}{q_{n_k}} \to 0.$$

It follows that, for an arbitrary $\rho \in (0, \eta)$, $\mathcal{E}(\Phi)$ contains a vector $(\rho_1, \rho_2, ..., \rho_d) \in \mathcal{E}(\Phi)$, with $\rho_{j_0} = \rho$. It follows that $\mathcal{E}(\Phi)$ includes a non-discrete subgroup of $\mathbb{R}^d$. □

Remark 2.3.3. Lemma 2.3.3 and Proposition 2.3.7 show that, in dimension 1, for $\Phi = 1_{[0,\beta]} - \beta$, if $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$, the group $\mathcal{E}(\Phi)$ contains at least a positive integer.

2.3.3 Well separated discontinuities, clusters of discontinuities

The previous method was based on Diophantine properties of the values of the integrals for rational cocycles (c.f. (2.3.1)). In this subsection we present results relying on diophantine properties of the discontinuities of the cocycle. We give sufficient conditions for regularity of the cocycle defined by a step function $\Phi : \mathbb{T} \to \mathbb{R}^d$ with integral 0.

The set of discontinuities of $\Phi_n(x) = \sum_{k=0}^{n-1} \Phi(x+k\alpha)$ is $\mathcal{D}_n := \Big\{\{x_i - k\alpha\} : 1 \leq i \leq D, 0 \leq k < n\Big\}$. We assume that the points $x_i - k\alpha \mod 1$, for $1 \leq i \leq D, 0 \leq k < n$, are distinct. The jump of Φ at x_i is $\sigma_i = \sigma(x_i) = \Phi(x_i^+) - \Phi(x_i^-)$. A discontinuity of the form $\{x_i - k\alpha\}$ is said to be of type x_i.

By Lemma 2.2.1, any interval of the circle of length $\geq 2/q_n$ contains at least one point of the set $\Big\{\{x_i - k\alpha\}, k = 0, \dots, q_n - 1\Big\}$, hence at least one discontinuity of Φ_{q_n} of type x_i for each $x_i \in \mathcal{D}$.

Well separated discontinuities

We write $\mathcal{D}_n = \{\gamma_{n,1} < \dots < \gamma_{n,Dn} < 1\}$ and $\gamma_{n,Dn+1} = \gamma_{n,1}$, where, for $1 \leq \ell \leq Dn$, the points $\gamma_{n,\ell}$ run through the set of discontinuities $\mathcal{D}_n$ in the natural order.

Definition 2.3.3. The cocycle is said to have *well separated discontinuities (wsd)*, if there is $c > 0$ and an infinite set $\mathcal{Q}$ of denominators of α such that

$$\gamma_{q,\ell+1} - \gamma_{q,\ell} \geq c/q, \ \forall q \in \mathcal{Q}, \ \ell \in \{1, \dots, Dq\}. \tag{2.3.11}$$

This condition is similar to Boshernitzan's condition ([5]) for interval exchange transformations. The result below extends an analogous statement when Φ takes values in $\mathbb{Z}^d$ (see [9]).

Theorem 2.3.8. *Let Φ be a zero mean step function. If Φ satisfies the wsd property (2.3.11), then the group $\mathcal{E}(\Phi)$ includes the set $\{\sigma_i : i = 1, \dots, D\}$ of jumps at discontinuities of Φ. Moreover, Φ is regular.*

Proof. Let us consider $\Phi_q(x)$ for $q \in \mathcal{Q}$. By (2.3.2) and (2.3.9), we can write, with $u_{(q)}^{i,j}(x)$ in a finite fixed set of integers $\mathcal{F}$,

$$\Phi_q = (\varphi_q^j)_{j=1,\dots,d} \text{ with } \varphi_q^j(x) = \sum_i t_{i,j}\, u_{(q)}^{i,j}(x) - \sum_i t_{i,j}\, \{q\beta_{i,j}\}.$$

Let $\theta^{(q)} := (\theta_{j,q}, j = 1, ..., d)$ with $\theta_{j,q} := -\sum_i t_{i,j} \{q\beta_{i,j}\}$. We can assume that the limit $\theta := \lim_{q\to\infty,\, q\in\mathcal{Q}} \theta^{(q)}$ exists. The set of values of Φ_q for $q \in \mathcal{Q}$ is included in $R + \theta^{(q)}$ where R is the finite fixed set of vectors $\{(\sum_i t_{i,j} k_{i,j}, j = 1, ..., d) :\ k_{i,j} \in \mathcal{F}\}$.

Let $\mathcal{I}_q$ be the partition of the circle into the intervals of continuity of Φ_q, $I_{q,\ell} = [\gamma_{q,\ell}, \gamma_{q,\ell+1})$, $1 \le \ell \le Dq$. With the constant c introduced in (2.3.11), let $J_{q,\ell} \subset \mathbb{T}$ be the union of $L := \lfloor 2/c \rfloor + 1$ consecutive intervals in $\mathcal{I}_q$ starting with $I_{q,\ell}$. By (2.3.11) every $J_{q,\ell}$ has length $\ge 2/q$, thus contains an element of the set $\Big\{\{x_i - s\alpha\}, s = 0, \ldots, q-1\Big\}$ for each x_i.

Therefore, for every jump σ_i of Φ, there is $v \in R$ and two consecutive intervals $I, I' \in \mathcal{I}_q$, with $I \cup I' \subset J_{q,\ell}$, such that the value of Φ_q is $v + \theta^{(q)}$ on I and $v + \theta^{(q)} + \sigma_i$ on I'.

Given $i \in \{1, ..., D\}$, we denote $\mathcal{H}_q(\sigma_i)$ the family of intervals $I \in \mathcal{I}_q$ such that the jump of Φ_q at the right endpoint of I is σ_i. Since each interval $J_{q,\ell}$ contains an interval $I \in \mathcal{H}_q(\sigma_i)$, the cardinality of $\mathcal{H}_q(\sigma_i)$ is at least $\frac{qD}{L}$.

Fix additionally $v \in R$ and let $\mathcal{A}_q(\sigma_i, v)$ be the set of intervals $I \in \mathcal{H}_q(\sigma_i)$ such that $\Phi_q(x) = v + \theta^{(q)}$ on I. Let $\mathcal{A}'_q(\sigma_i, v)$ be the set of intervals $I' \in \mathcal{I}_q$ adjacent on the right to the intervals $I \in \mathcal{A}_q(\sigma_i, v)$.

Let $A_q(\sigma_i, v)$ be the union of intervals $I \in \mathcal{A}_q(\sigma_i, v)$ and $A'_q(\sigma_i, v)$ the union of intervals $I' \in \mathcal{A}'_q(\sigma_i, v)$. The value of Φ_q is $v + \theta^{(q)}$ on $A_q(\sigma_i, v)$ and $v + \theta^{(q)} + \sigma_i$ on $A'_q(\sigma_i, v)$.

There is $v_0 \in R$ and an infinite subset $\mathcal{Q}_0$ of $\mathcal{Q}$ such that, for $q \in \mathcal{Q}_0$,

$$|\mathcal{A}_q(\sigma_i, v_0)|,\ |\mathcal{A}'_q(\sigma_i, v_0)| \ge \frac{|\mathcal{H}_q(\sigma_i)|}{|R|} \ge \frac{qD}{L|R|}. \tag{2.3.12}$$

By (2.3.11) and (2.3.12), we have $\mu\left(A_q(\sigma_i, v_0)\right),\ \mu\left(A'_q(\sigma_i, v_0)\right) \ge \frac{Dc^2}{(2+c)|R|}$. Thus $v_0 + \theta$ and $v_0 + \theta + \sigma_i$ are quasi-periods, hence, by Lemma 2.1.5 essential values. Since $\mathcal{E}(\Phi)$ is a group, σ_i is an essential value. Therefore $\mathcal{E}(\Phi)$ includes the group generated by the jumps of Φ.

Finally, notice that the quotient cocycle $\Phi/\mathcal{E}(\Phi)$ is a continuous step cocycle, hence is constant. Therefore, the regularity of Φ follows from Lemma 2.1.10. □

For $\Phi_d := \Big(\mathbb{1}_{[0,\beta_j]} - \beta_j,\ j = 1, ..., d\Big)$ with $\beta_i \neq \beta_j$ whenever $i \neq j$, the jump of Φ_d is $(1, ..., 1)$ at 0 and $(0, ..., 0, -1, 0, ..., 0)$ at β_j (-1 stands at the j-th coordinate), $j = 1, \ldots, d$. If the wsd property is satisfied, the group $\mathcal{E}(\Phi)$ includes $\mathbb{Z}^d$. Therefore the cocycle Φ_d is regular whenever the wsd property holds.

In view of Lemma 2.2.3 and Theorem 2.3.8, we obtain the following result (where the case $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ can be treated directly).

Corollary 2.3.9. *Let α be of bounded type. Let β be a real number.*

1) The cocycle $(\mathbb{1}_{[0,\frac{r}{s})}(.) - \frac{r}{s},\ \mathbb{1}_{[0,\frac{r}{s})}(.+\beta) - \frac{r}{s})$ is regular for every rational number $\frac{r}{s} \in (0,1)$.

2) If $\frac{r_1}{s_1}, ..., \frac{r_d}{s_d}$ are rational numbers such that $0 < \frac{r_i}{s_i}\beta < 1$, then, for every real numbers $t_1, ..., t_d$, the cocycle $\varphi = \sum_i t_i \mathbb{1}_{[0,\frac{r_i}{s_i}\beta)} - \beta \sum_i t_i \frac{r_i}{s_i}$ is regular.

Clusters of discontinuities

For a subset C of discontinuities of Φ, we denote $\sigma(C) = \sum_{x_i \in C} \sigma(x_i)$ the corresponding sum of jumps of Φ. The number of discontinuities of Φ is $D = D(\Phi)$. The following result can be useful when the discontinuities are not well separated.

Theorem 2.3.10. *Suppose that there are two discontinuities x_{i_0}, x_{j_0} of Φ and a subsequence (q_{n_k}) such that for a constant $\kappa > 0$ we have*

$$q_{n_k} \|(x_{i_0} - x_{j_0}) - r\alpha\| \geq \kappa, \ \forall\ |r| < q_{n_k}. \tag{2.3.13}$$

Then, if the sum $\sigma(C)$ is $\neq 0$ for each non-empty proper subset C of the set of discontinuities of Φ, then Φ has a non trivial essential value.

Proof. By Lemma 2.2.1 any interval of length $2/q_n$ on the circle contains at least one discontinuity of each type x_i and at most 4 such discontinuities, therefore at most $4D(\Phi)$ discontinuities of Φ_{q_n}.

Consider the sequence $\mathcal{Q} = (q_{n_k})$ of denominators satisfying (2.3.13). On the circle $\mathbb{T}$ we will deal with families of disjoint intervals of length $4/q_{n_k}$. In fact, we consider families of the form $\left\{I_j^{(k)} : j \in J_k \subset \{0, 1, ..., q_{n_k} - 1\}\right\}$ with $I_j^{(k)} = I_0^{(k)} + \{-j\alpha\}$, where $I_0^{(k)} = [0, 4/q_{n_k}]$ and J_k is such that its cardinality satisfies $|J_k| \geq \delta_1 q_{n_k}$ for a fixed positive constant δ_1.

The number of different "patterns of discontinuities" (i.e. consecutive types of discontinuities) which can occur altogether in these intervals is finite (indeed, the length of a pattern of discontinuity is bounded by $8D(\Phi)$). There are an infinite subsequence of $\mathcal{Q}$ (still denoted by $\mathcal{Q}$) and a family of intervals $I_0^{(k)} + \{-j\alpha\}$, $j \in J'_k$ with $|J'_k| \geq \delta_2 q_{n_k}$ for a fixed positive constant δ_2 (therefore with a total amount of measure bounded away from 0) such that the same pattern of discontinuities occurs in each interval of the family. For illustration, if the cocycle has 4 discontinuities x_1, x_2, x_3, x_4, we can have for instance in each interval the pattern $(x_1, x_3, x_4, x_3, x_2, x_1, x_2, x_4)$, corresponding in a given interval to the "configuration" (a sequence of discontinuities) of the form $(\{x_1-\ell_{j,1}\alpha\}, \{x_3-\ell_{j,2}\alpha\}, \{x_4-\ell_{j,3}\alpha\}, \{x_3-\ell_{j,4}\alpha\}, \{x_2-\ell_{j,5}\alpha\}, \{x_1-\ell_{j,6}\alpha\}, \{x_2-\ell_{j,7}\alpha\}, \{x_4-\ell_{j,8}\alpha\})$.

Now, by taking a further subsequence of $\mathcal{Q}$ if necessary, we will assure a convergence at scale $1/q_{n_k}$ for the discontinuities in $I_j^{(k)}$. More precisely, observe that if $\{x_i-\ell\alpha\} \in I_j^{(k)}$, then $\{x_i-\ell\alpha\}-\{-j\alpha\} \in I_0^{(k)}$. Hence $\{x_i-(\ell-j)\alpha\}$ is in $I_0^{(k)}$ and therefore it belongs to the set $\Big\{\{x_i-u\alpha\} : |u| < q_{n_k}\Big\} \cap I_0^{(k)}$. Notice that this set $\Big\{\{x_i-u\alpha\} : |u| < q_{n_k}\Big\} \cap I_0^{(k)}$ has at least 2 elements and has no more that 8 elements and it does not depend on j (when k changes, the set J_k' does and so j are different for different k, however the common shift, namely the shift by $j\alpha$, leads to points which will be common for all $j \in J_k'$; on the other hand r runs over a fixed set as the patterns of discontinuities are the same regardless k and j). Therefore we can write it explicitly as $\Big\{\{x_i-u_{n_k,i,r}\alpha\}\Big\}$.

We can extract a new subsequence of $\mathcal{Q}$ (for which we still keep the same notation $\mathcal{Q}=(q_{n_k})$) such that for each $\{x_i-u_{n_k,i,r}\alpha\}$ the sequence $q_{n_k}\{x_i-u_{n_k,i,r}\alpha\}$ converges to a limit $y_{i,r} \in [0,4]$ when $k \to \infty$. This is possible, since there is a finite number of such points in $I_0^{(k)}$ for each n_k.

Therefore the configurations of discontinuities in the intervals $I_j^{(k)}$ for $j \in J_k'$ are converging at the scale $1/q_{n_k}$, i.e. after applying the affinities $x \to q_{n_k}(x-\{-j\alpha\})$. We can group the discontinuities (of type) x_i according to the value of the limit $y_{i,r}$.

We call "clusters" the subsets of discontinuity points in $I_j^{(k)}$ with the same limit at the scale q_{n_k} (hence, such that the corresponding limits $y_{i,r}$ in $[0,4]$ coincide). Observe that two discontinuities of the same type x_i are at distance $\geq \frac{1}{2q_{n_k}}$ by the point 4) of Lemma 2.2.1 and therefore are not in the same cluster: a cluster contains at most one discontinuity of a given type x_i. In view of (2.3.13), the number of elements in a cluster is strictly less than $D(\Phi)$ the number of discontinuities of Φ.

By passing once more to a subsequence of $\mathcal{Q}$ (still denoted by $\mathcal{Q}=(q_{n_k})$) if necessary, we extract a sequence of families of disjoint "good" intervals of length $4/q_{n_k}$ with the same configuration of clusters inside the intervals of a family. There are at least three different clusters in each "good" interval (since for an interval of length $4/q_{n_k}$ a given type of discontinuity occurs at least twice and must occur in different clusters as shown above, moreover the number of elements in a cluster is at most $D(\Phi)-1$). The clusters in each interval are separated by more than c/q_{n_k}. As in the proof of Theorem 2.3.8, the values of the cocycle at time q_{n_k} are $v+\theta^{(q_{n_k})}$ with v in a fixed finite set and $(\theta^{(q_{n_k})})$ a converging sequence.

For k large, clusters of discontinuities are separated by intervals of order c_1/q_{n_k} for a fixed positive constant c_1 and there are at least 3 clusters in a "good" interval $I_j^{(k)}$. The number of intervals in the families is greater than a fixed fraction of q_{n_k}. It follows that, under the assumption that the sum of jumps $\sigma(C)$ is $\neq 0$ for each non-empty proper subset C of the set of discontinuities of Φ, the cocycle at time q_{n_k} is close to a non zero constant on a set which has a measure bounded away from 0.

Therefore there Φ has a non trivial quasi-period, hence a non trivial finite essential value. □

Examples of application of Theorem 2.3.10

Recall that by Remark 2.3.1, if $x_1, \dots, x_D$ are all discontinuities of a step cocycle Φ, then for $i \neq j$ we can assume that $x_i - x_j$ is not a multiple of α modulo 1. Assume that α is of bounded type. Then, fixing $i_0 \neq j_0$ and using Lemma 2.2.3 to select a subsequence (q_{n_k}) so that (2.3.13) holds for a constant $\kappa > 0$, the assumption of the theorem are fulfilled.

Example 1: cocycle with 3 discontinuities

Let α be an irrational number *of bounded type*. Let φ be a scalar cocycle with 3 effective discontinuities $0, \beta, \gamma$. The sum of jumps for the 3 discontinuities is 0, and for subsets of 1 or of 2 discontinuities it is always non zero. If β (resp. γ) is not in $\mathbb{Z}\alpha + \mathbb{Z}$, by Lemma 2.2.3 there are subsequences of denominators along which the discontinuities of type β (resp. γ) belong to clusters which reduce to a single discontinuity or to two discontinuities. Therefore, by Theorem 2.3.10 the group of finite essential values does not reduce to $\{0\}$.

Example 2: cocycle with 4 discontinuities

Let us consider the $\mathbb{R}$-valued cocycle $a(\mathbb{1}_{[0,\beta)}(\cdot) - \beta) - (\mathbb{1}_{[0,\beta)}(\cdot - \gamma) - \beta)$ with $\beta < \gamma$.

There are 4 discontinuity points: $(0, \beta, \gamma, \beta + \gamma)$ with respective jumps $+a$, $-a$, -1, $+1$.

Assume that β is such that there is a subsequence (q_{n_k}) and a constant $\kappa > 0$ such that

$$q_{n_k} \|\beta - r\alpha\| \geq \kappa, \ \forall \ |r| < q_{n_k}. \tag{2.3.14}$$

We apply the method of Theorem 2.3.10, with the subsequence (q_{n_k}). By the above condition on β, in a cluster we can find either a single discontinuity, or two discontinuities of type in $(0, \gamma)$, $(0, \beta + \gamma)$, $(\beta, \gamma), (\beta, \beta + \gamma)$ with respective sum of jumps: $a - 1$, $a + 1$, $-(a + 1)$, $-a + 1$. The case of 3 discontinuities is excluded.

Therefore, if $a \notin \{\pm 1\}$, we have a non trivial essential value. When $a = -1$, then the cocycle reads $-\mathbb{1}_{[0,\beta)}(\cdot) - \mathbb{1}_{[0,\beta)}(\cdot - \gamma) + 2\beta$, and by Theorem 2.3.5 or the method of Proposition 2.3.7 we obtain a non trivial essential value.

So for the classification of the cocycle $a(\mathbb{1}_{[0,\beta)}(\cdot) - \beta) - (\mathbb{1}_{[0,\beta)}(\cdot - \gamma) - \beta)$ the only case to be considered is $a = 1$. This leaves open the question of the regularity of the cocycle $\mathbb{1}_{[0,\beta)}(\cdot) - (\mathbb{1}_{[0,\beta)}(\cdot - \gamma)$ for α of bounded type and any β, γ.

We would like to mention that when $\beta = 1/2$ and α is of bounded type the regularity (for each $\gamma \in \mathbb{T}$) has been shown recently by Zhang [52] using different methods. In fact, Zhang shows that the cocycle $\Phi = (\mathbb{1}_{[0,1/2)}(\cdot) - 1/2, \mathbb{1}_{[0,1/2)}(\cdot + \gamma) - 1/2)$ is regular (whenever α is of bounded type).

The regularity of Φ follows also from Lemma 2.2.3 and Theorem 2.3.8 (see Corollary 2.3.9).

Example 3 The method of Theorem can be applied to the lower dimensional cocycle: $\varphi = \mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\gamma)} - \beta + \gamma$ when $(1, \alpha, \beta, \gamma)$ are rationally dependent.

2.3.4 On the regularity of Φ_d, $d = 1, 2, 3$

d= 1, $\Phi_1 = \mathbb{1}_{[0,\beta)} - \beta$

Theorem 2.3.11. *The cocycle $\Phi_\beta = \mathbb{1}_{[0,\beta)} - \beta$ is regular over any irrational rotation.*

Proof. If $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$, then Φ_β is a coboundary (see Remark 2.3.1). Suppose that $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$. Then, by Lemma 2.3.3 and Proposition 2.3.7, there is a positive integer in the group $\mathcal{E}(\Phi)$ (cf. Remark 2.3.3). Therefore the cocycle Φ_β is always regular. □

Remark 2.3.4. If $\beta, \alpha, 1$ are independent over $\mathbb{Q}$, then by a result of Oren ([37]) the cocycle defined by Φ_β is ergodic.

d= 2, $\Phi_2 = (\mathbb{1}_{[0,\beta)} - \beta, \mathbb{1}_{[0,\gamma)} - \gamma)$

a) α **of bounded type**

Theorem 2.3.12. *If α is of bounded type, the cocycle $\Phi_2 = (\mathbb{1}_{[0,\beta)} - \beta, \mathbb{1}_{[0,\gamma)} - \gamma)$ is regular.*

Proof. Recall that we constantly assume that $\beta, \gamma, \beta - \gamma$ are not in $\mathbb{Z}\alpha + \mathbb{Z}$. The proof is done in three steps:

Step 1. $\mathcal{E}(\Phi_2) \neq \{0\}$; indeed, this follows immediately from the proof of Theorem 2.3.4 applied to $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ (see Lemma 2.2.4 and Lemma 2.3.3).

Step 2. $\beta = \gamma$; then our cocycle is regular by Theorem 2.3.11.

Step 3. $0 < \beta < \gamma < 1$. Now, we claim that for each $a, b \in \mathbb{R}$ the cocycle $a(\mathbb{1}_{[0,\beta)} - \beta) + b(\mathbb{1}_{[0,\gamma)} - \gamma)$ is regular. Indeed, we have already noticed this property to hold if a or b is equal to zero. When $a \neq 0 \neq b$, we obtain a step cocycle with 3 effective discontinuities $0, \beta$ and γ. In that case we apply Theorem 2.3.10 (see the application to cocycles with 3 discontinuities, example 1 after the proof) to conclude that our scalar cocycle has a non-zero finite essential value and hence is regular. The claim immediately follows. The regularity of Φ is now an immediate consequence of Corollary 2.1.9. □

Remark 2.3.5. Notice that we can apply other previous results to obtain another, more complex proof of Theorem 2.3.12, which however can be applied in other situations. Indeed, since α of bounded type, we apply Theorem 2.3.5 to conclude that the cocycle Φ is regular whenever $\beta, \gamma, \alpha, 1$ are independent over $\mathbb{Q}$.

Otherwise, there are integers r, s, v, w not all equal to zero such that

$$r\beta + s\gamma + v\alpha + w = 0.$$

The case when β or γ belongs to $\mathbb{Z}\alpha + \mathbb{Z}$ is excluded (cf. Remark 2.3.1).

1) Assume that $\beta, \gamma \notin \mathbb{Q}\alpha + \mathbb{Q}$ and $\beta - \gamma \notin \mathbb{Q}\alpha + \mathbb{Q}$.

If s or $r \neq 0$, say $s \neq 0$ then $\gamma = -\frac{r}{s}\beta - \frac{v}{s}\alpha - \frac{w}{s}$. We apply Lemma 2.2.3 for $\beta_1 = \frac{1}{1}\beta + \frac{0}{1}\alpha + \frac{0}{1}$, $\beta_2 = \frac{-r}{s}\beta + \frac{-v}{s}\alpha + \frac{-w}{s}$ and $\beta_3 = \frac{-r-s}{s}\beta + \frac{-v}{s}\alpha + \frac{-w}{s}$ and obtain a subsequence (q_{n_k}) along which the wsd property is satisfied for the discontinuities of Φ_2. Then Theorem 2.3.8 applies.

2) Suppose $s = 0$ and $\gamma \notin \mathbb{Q}\alpha + \mathbb{Q}$, $\beta \in \mathbb{Q}\alpha + \mathbb{Q}$. It is enough to show that $d_1 = 2$ in Theorem 2.3.4. By the proof of that theorem applied to $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$, in view of Lemma 2.2.4, we obtain $M : \mathbb{R}^2 \to \mathbb{R}^2$ a rational change of coordinates such that $M\Phi_2 = (\psi^1, \psi^2)$ has $(1, 0)$ as its essential value. On the other hand, by Lemma 2.3.2 (taking into account that $\det M \neq 0$) and remembering that under our assumption β and γ are independent over $\mathbb{Q}$, we obtain that $\beta(\psi^i) \notin \mathbb{Z}\alpha + \mathbb{Z}$, $i = 1, 2$. Therefore, again by Lemma 2.2.4, $\mathcal{L}(\beta(\psi^i)) \neq \{0\}$, hence by the proof of Theorem 2.3.4, $d_1 = 2$.

3) The missing case $\beta - \gamma \in \mathbb{Q}\alpha + \mathbb{Q}$ (see the assumption in 1) and the separate case $\beta, \gamma \in \mathbb{Q}\alpha + \mathbb{Q}$) are covered by Lemma 2.2.3 and an application of Theorem 2.3.8.

b) α **of non bounded type**

For $d = 2$ and α not of bounded type the question of construction of a non regular step function is not solved and the purpose of this paragraph is to present some observations.

From Lemma 2.3.3 and Proposition 2.3.7, we know that $\mathcal{E}(\Phi)$ does not reduce to $\{0\}$. By Corollary 2.1.9, the regularity of the cocycle is equivalent to the regularity of the one dimensional cocycles with 3 discontinuities: $\varphi = a(1_{[0,\beta)} - \beta) - b(1_{[0,\gamma)} - \gamma)$, where a, b are arbitrary real numbers. Since we know already that regularity holds for $b = 0$, it suffices to consider $\varphi = a1_{[0,\beta)} - 1_{[0,\gamma)} - (a\beta - \gamma)$. It is interesting to understand the particular case $\gamma = \ell\beta$, with ℓ a positive integer. We will give some partial results on this cocycle and ask questions.

First of all, there are special situations where one can conclude that the cocycle $\varphi = \ell\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\ell\beta)}$ is a coboundary (we assume that $\ell\beta < 1$). We use the following result of Guenais and Parreau (with the notation of Section **??**, in particular $Tx = x + \alpha$):

Theorem 2.3.13. *([16], Theorem 2) Let φ be a step function on $\mathbb{T}$ with integral 0 and jumps $-s_j$ at distinct points $(\beta_j, 0 \le j \le m)$, $m \ge 1$, and let $t \in \mathbb{T}$. Suppose that there is a partition $\mathcal{P}$ of $\{0, \dots, m\}$ such that for every $J \in \mathcal{P}$ and $\beta_J \in \{\beta_j : j \in J\}$:*
(i) $\sum_{j\in J} s_j \in \mathbb{Z}$;
(ii) for every $j \in J$, there is a sequence of integers $(b_n^j)_n$ such that

$$\beta_j = \beta_J + \sum_{n\ge0} b_n^j q_n\alpha \mod 1, \text{ with } \sum_{n\ge0} \frac{|b_n^j|}{a_{n+1}} < +\infty \text{ and } \sum_{n\ge0} \Big\|\sum_{j\in J} b_n^j s_j\Big\|^2 < +\infty;$$

(iii) there is an integer k' such that $t = k'\alpha - \sum_{J\in\mathcal{P}} t_J$ where

$$t_J = \beta_J \sum_{j\in J} s_j + \sum_{n\ge0} \Big[\sum_{j\in J} b_n^j s_j\Big] q_n\alpha \mod 1.$$

Then there is a measurable function f of modulus 1 solution of

$$e^{2i\pi\varphi} = e^{2i\pi t} f \circ T/f. \tag{2.3.15}$$

Conversely, when $\sum_{j\in J} s_j \notin \mathbb{Z}$ for every proper non empty subset J of $\{0, .., m\}$, these conditions are necessary for the existence of a solution of (2.3.15).

Take $\varphi = \ell\mathbb{1}_{[0,\beta]} - \mathbb{1}_{[0,\ell\beta]}$. With the previous notation, the discontinuities are at $\beta_0 = 0, \beta_1 = \beta, \beta_2 = \gamma = \ell\beta$ ($m = 2$) with jumps $\ell - 1, -\ell, 1$ respectively and the partition $\mathcal{P}$ is the trivial partition with the single atom $J = \{0, 1, 2\}$. We also have $\beta_J = 0$, $\sum_{j\in J} s_j = 0$.

Suppose that the parameter β has an expansion in base $(q_n\alpha)$ (Ostrowski expansion, see [18]):

$$\beta = \sum_{n\geq 0} b_n q_n \alpha \mod 1, \text{ with } \sum_{n\geq 0} \frac{|b_n|}{a_{n+1}} < +\infty, \; b_n \in \mathbb{Z}. \tag{2.3.16}$$

We can take $b_n^0 = 0, b_n^1 = b_n, b_n^2 = \ell b_n$, so that $\sum_{j\in J} b_n^j s_j = \ell b_n - \ell b_n = 0$. In view of Theorem 2.3.13, for every real s, the multiplicative equation $e^{2\pi i s\varphi} = f \circ T/f$ has a measurable solution $f : \mathbb{T} \to \mathbb{S}^1$. By using Theorem 6.2 in [35], we conclude that φ is a measurable coboundary. Let us mention that another proof based on the tightness of the cocycle (φ_n) can also be given.

Conversely, if φ is a measurable coboundary, then $e^{2\pi i s\varphi} = f \circ T/f$, for s real has a measurable solution, and this implies that β has the expansion given by (2.3.16).

Therefore we obtain:

Proposition 2.3.14. *If ℓ is a positive integer with $\ell\beta < 1$, then the cocycle $\varphi = \ell\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\ell\beta)}$ is a coboundary if and only if β satisfies (2.3.16).*

Question: A question is to know if the cocycle $\varphi = \ell\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\ell\beta)}$ is regular or not, when β has an expansion $\beta = \sum_{n\geq 0} b_n q_n \alpha \mod 1$, with $\lim_n \frac{|b_n|}{a_{n+1}} = 0$ and $\sum_{n\geq 0} \frac{|b_n|}{a_{n+1}} = +\infty$. (Notice that by Theorem 2.3.13 it cannot be a coboundary.)

$$\mathbf{d= 3,}\ \Phi_3 = (\mathbb{1}_{[0,\beta)} - \beta, \mathbb{1}_{[0,\gamma)} - \gamma, \mathbb{1}_{[0,\delta)} - \delta)$$

We will consider α of non bounded type and show that for some choice of β, γ, δ we can obtain a non regular cocycles (cf. [7]). For $r \in \mathbb{R}$, we denote by ρ_r the translation $x \to x + r \mod 1$.

Theorem 2.3.15. *Assume that $Tx = x + \alpha$ on the circle $\mathbb{T}$. If α is not of bounded type, then there exists β such that $\varphi = \mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\beta)} \circ \rho_r$ is a non regular cocycle for r in a set of full Lebesgue measure.*

Proof. By a result of Merril ([34], Theorem 2.5 therein, see also Theorem 2.3.13 above from [16]), we know that, if β satisfies (2.3.16), then there is an uncountable set of real numbers s (so containing irrational numbers) such that we can solve the following quasi-coboundary multiplicative equation in (s, β): for $s \in \mathbb{R}$ there exist $|c| = 1$ and a measurable function $f : \mathbb{T} \to \mathbb{S}^1$ such that $e^{2\pi i s 1_{[0,\beta)}} = cf/f \circ T$.

For this choice of β and s (s is irrational), $e^{2\pi i s(1_{[0,\beta)} - 1_{[0,\beta)} \circ \rho_r)}$ is a multiplicative coboundary for every r.

For the integer valued cocycle $\psi_r = 1_{[0,\beta)} - 1_{[0,\beta)} \circ \rho_r$ we obviously have $\mathcal{E}(\psi_r) \subset \mathbb{Z}$. On the other hand, $s\,\psi_r(x) = n(x) + F(x) - F(x+\alpha)$, with $F : X \to \mathbb{R}$ and $n(\cdot) : X \to \mathbb{Z}$ measurable. Therefore $\psi_r(x) = s^{-1}n(x) + s^{-1}F(x) - s^{-1}F(x+\alpha)$. It follows that the group of finite essential values over T of the cocycle ψ_r is also included in the group $\frac{1}{s}\mathbb{Z}$ and therefore $\overline{\mathcal{E}}(\psi_r) \subset \{0, \infty\}$.

This implies that ψ_r is either non regular or a coboundary (cf. Subsection 2.1.1). The latter case cannot occur for a set of values of r of positive measure, because otherwise, by Proposition 2.3.16 below, $\mathbb{1}_{[0,\beta)} - \beta$ is an additive coboundary up to some additive constant c (and necessarily $c = 0$, since the cocycle defined by $\mathbb{1}_{[0,\beta)} - \beta$ is recurrent). But this would imply that $e^{2\pi i\beta}$ is an eigenvalue of the rotation by α, a contradiction.

Therefore the cocycle $\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\beta)} \circ \rho_r$ is non regular for a.e. $r \in \mathbb{R}$. □

Proposition 2.3.16. *Assume that K is a compact connected Abelian (monothetic) group. Let T be an ergodic rotation on K. Let $\varphi : K \to \mathbb{R}$ be a cocycle. Assume moreover, than on a set of $g \in K$ of positive Haar measure we can find a measurable function $\psi_g : K \to \mathbb{R}$ such that*

$$\varphi - \varphi(g + \cdot) = \psi_g \circ T - \psi_g. \tag{2.3.17}$$

Then φ is an additive quasi-coboundary, i.e. $\varphi = b + h \circ T - h$, for a measurable function $h : K \to \mathbb{R}$ and a constant $b \in \mathbb{R}$.

Proof. For $g \in K$ satisfying (2.3.17) and arbitrary $s \in \mathbb{R}$ we have:

$$\frac{e^{2\pi i s\varphi(x)}}{e^{2\pi i s\varphi(g+x)}} = \frac{e^{2\pi i s\psi_g(Tx)}}{e^{2\pi i s\psi_g(x)}}.$$

According to Proposition 3 in [32], for every s there exist λ_s with $|\lambda_s| = 1$ and a measurable function $\zeta_s : X \to \mathbb{S}^1$ such that $e^{2\pi i s\varphi} = \lambda_s \cdot \zeta_s \circ T/\zeta_s$. By Theorem 6.2 in [35], the result follows. □

Remark 2.3.6. 1) If β satisfies (2.3.16), then either $\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\beta)} \circ \rho_r$ is non regular or is a coboundary. We have shown that the latter case can occur only for r in a set of zero measure. A problem is to explicit values of r for which $\mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\beta)} \circ \rho_r$ is not a coboundary.

2) If $\psi_{\beta,\frac{1}{2}} := \mathbb{1}_{[0,\beta)} - \mathbb{1}_{[0,\beta)} \circ \rho_{\frac{1}{2}}$ is non regular, then $\psi_{[\frac{1}{2}-\beta,\frac{1}{2})} := \mathbb{1}_{[0,\frac{1}{2}-\beta)} - \mathbb{1}_{[0,\frac{1}{2}-\beta)} \circ \rho_{\frac{1}{2}}$ is regular. Indeed the sum of these two cocycles is $\mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)}$. It can be easily shown that this latter cocycle has non trivial quasi periods. The non regularity of $\psi_{\beta,\frac{1}{2}}$ implies that $(\psi_{[\beta,\frac{1}{2})})_{q_n}$, the cocycle at times q_n, tends to 0 in probability, so that $\psi_{[\frac{1}{2}-\beta,\frac{1}{2})}$ has non trivial quasi periods.

Corollary 2.3.17. *There are values of the parameters* (β, γ, δ) *such that*

$$\Phi_3 = (\mathbb{1}_{[0,\beta)} - \beta, \mathbb{1}_{[0,\gamma)} - \gamma, \mathbb{1}_{[0,\delta)} - \delta)$$

is non regular.

Proof. Suppose that $0 < \beta < \gamma < \delta$ and $\delta = \beta + \gamma$. By applying the map $(y_1, y_2, y_3) \to y_1 + y_2 - y_3$, we obtain the 1-dimensional cocycle $\mathbb{1}_{[0,\beta)}(\cdot) - \mathbb{1}_{[0,\beta)}(\cdot + \gamma)$, which is non regular by Theorem 2.3.15 for a value of the parameter β satisfying (2.3.16) and almost all γ. Lemma 2.1.3 implies the non regularity of Φ_3 for these values of the parameters. □

Note that for $d = 2$, i.e. for two parameters (β, γ), an attempt to obtain a non regular cocycle is to take $\gamma = 2\beta$ and the linear combination: $2(\mathbb{1}_{[0,\beta)}(\cdot) - \beta) - (\mathbb{1}_{[0,2\beta)}(\cdot) - 2\beta) = \mathbb{1}_{[0,\beta)}(\cdot) - \mathbb{1}_{[0,\beta)}(\cdot + \beta)$. We obtain the cocycle discussed above (cf. Proposition 2.3.14) and the question previously mentioned above is whether there are values of β such that it is non regular.

2.4 Application to affine cocycles

We consider now the affine cocycle

$$\Psi_{d+1}(x) := (\psi(x), \psi(x + \beta_1), ..., \psi(x + \beta_d)), \text{ where } \psi(x) = \{x\} - \frac{1}{2}.$$

2.4.1 Reduction to a step function

By a straightforward calculation we have the following formula for the cocycle ψ:

$$\psi_{q_n}(x) = q_n x + \frac{q_n(q_n - 1)}{2}\alpha - \frac{q_n}{2} + M(x), \tag{2.4.1}$$

where M is a (non 1-periodic) function with values in $\mathbb{Z}$. It follows that, for $\beta \in [0, 1)$,

$$\psi_{q_n}(\{x + \beta\}) = \tag{2.4.2}$$

$$= \begin{cases} \psi_{q_n}(x) + q_n\beta + (M(x + \beta) - M(x)) & \text{if} \quad x + \beta < 1, \\ \psi_{q_n}(x) + (q_n\beta - q_n) + (M(\{x + \beta\}) - M(x)) & \text{if} \quad 1 \leqslant x + \beta < 2. \end{cases}$$

We will reduce the cocycle Ψ_{d+1} to step cocycles using the group of finite essential values.

Theorem 2.4.1. *The group* $\mathcal{E}(\Psi_{d+1})$ *includes* $\Delta_{d+1} = \{(t, ..., t) : t \in \mathbb{R}\}$, *the diagonal subgroup of* $\mathbb{R}^{d+1}$.

Proof. Denote $S_i(x) = \rho_{\beta_i}(x) = x + \beta_i \mod 1$. Suppose that $\{q_{n_k}\beta_i\} \to c_i$, with $c_i \in [0, 1)$ for $i = 1, \ldots, d$, and consider the measures

$$\nu_k := ((\psi \times \psi \circ S_1 \times \ldots \times \psi \circ S_d)_{q_{n_k}})_*(\mu), \quad k \geq 1.$$

Since

$$\forall x, y \in [0, 1), \quad |\psi_{q_{n_k}}(x) - \psi_{q_{n_k}}(y)| < 2\,V(\psi) = 2$$

and $\int \psi\, d\mu = 0$, we have that $\mathrm{Im}(\psi \times \psi \circ S_1 \times \ldots \times \psi \circ S_d)_{q_{n_k}} \subset [-2, 2]^{d+1}$, so that ν_k is concentrated on $[-2, 2]^{d+1}$.

It follows that we can select a subsequence of (ν_k) (still denoted (ν_k)) which converges to a probability measure ν (which is concentrated on $[-2, 2]^{d+1}$). We will show in what kind of a subset of $\mathbb{R}^{d+1}$ the support of ν is included. Consider the image of the measure ν_k via

$$F : \mathbb{R}^{d+1} \to \mathbb{R}^d, \quad F(x_0, \ldots, x_d) = (x_1 - x_0, \ldots, x_d - x_0).$$

In view of (2.4.2), we obtain

$$F \circ (\psi \times \psi \circ S_1 \times \ldots \times \psi \circ S_d)_{q_{n_k}}(x) = (\{q_{n_k}\beta_1\} + M_1(x), \ldots, \{q_{n_k}\beta_d\} + M_d(x))$$

with $M_i(x) \in \mathbb{Z}$, whence $F_*\nu_k$ is the measure concentrated on $(\{q_{n_k}\beta_1\}, ..., \{q_{n_k}\beta_d\}) + \mathbb{Z}^d$.

Since $\nu_k \to \nu$ weakly, $F_*\nu_k \to F_*\nu$ (because all these measures are concentrated on a bounded subset of $\mathbb{R}^{d+1}$). As $\{q_{n_k}\beta_i\} \to c_i$, it follows that

$$\operatorname{supp} \nu \subset \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} : \ x_i - x_0 = c_i + k_i, \ k_i \in \mathbb{Z}, \ i = 1, \ldots, d\}.$$

The set on the right hand side of this inclusion is equal to the union of sets of the form $\{(x, x-(c_1+k_1), ..., x-(c_d+k_d) : x \in \mathbb{R}\}$, hence of countably many lines parallel to the diagonal Δ_{d+1}. Moreover, the support of ν is uncountable (because one dimensional projections of ν are absolutely continuous measures - see [30]), whence it must be uncountable on one of these lines. In view of Proposition 2.1.4, $\operatorname{supp} \nu \subset \mathcal{E}(\Psi_{d+1})$ and since $\mathcal{E}(\Psi_{d+1})$ is a group, we have $\operatorname{supp} \nu - \operatorname{supp} \nu \subset \mathcal{E}(\Psi_{d+1})$. However, the set $\Delta_{d+1} \cap (\operatorname{supp} \nu - \operatorname{supp} \nu)$ is uncountable, so because $\mathcal{E}(\Psi_{d+1})$ is closed, we must have $\Delta_{d+1} \subset \mathcal{E}(\Psi_{d+1})$ and the proof is complete. □

Corollary 2.4.2. *$(\psi, \psi \circ S_1, \ldots, \psi \circ S_d)$ is ergodic whenever the set of accumulation points of $(\{q_n\beta_1\}, \ldots, \{q_n\beta_d\})$ is dense in $\mathbb{T}^d$.*

Proof. From the proof of Proposition 2.4.1 it follows that with every accumulation point $(c_1, \ldots, c_d)$ of $(\{q_n\beta_1\}, \ldots, \{q_n\beta_d\})$ we obtain a line $\{(x, x - (c_1 + k_1), ..., x - (c_d + k_d) : x \in \mathbb{R}\}$ (and the smallest subgroup in which the line is included) which is included in the group of essential values. Since the set of accumulation points is dense and $\mathcal{E}(\Psi_{d+1})$ is closed, it follows that the only possibility is that $\mathcal{E}(\Psi_{d+1}) = \mathbb{R}^{d+1}$ which is equivalent to the fact that Ψ_{d+1} is ergodic. □

By Lemma 2.1.2 the study of Ψ_{d+1} reduces to that of the quotient cocycle $\Psi_{d+1} + \Delta_{d+1} : \mathbb{T} \to \mathbb{R}^{d+1}/\Delta_{d+1}$. Using the epimorphism $\mathbb{R}^{d+1} \ni (y_0, ..., y_d) \to (y_1 - y_0, ..., y_d - y_0) \in \mathbb{R}^d$ (whose kernel is equal to Δ_{d+1}), the quotient is given by the cocycle

$$\Phi_d(x) = (\mathbb{1}_{[0,1-\beta_j)} - 1 + \beta_j)_{j=1,\ldots,d}. \tag{2.4.3}$$

2.4.2 On the regularity of Ψ_{d+1}, $d = 1, 2, 3$.

1) $d = 1, \Psi_2 = (\psi(x), \psi(x + \beta))$

Applying Theorem 2.4.1 and the equation 2.4.3 we can reduce the cocycle Ψ_2 to the quotient cocycle $(\Psi_2 + \Delta_2)(x) = \mathbb{1}_{[0,1-\beta)} - 1 + \beta$. We conclude using Theorem 2.3.11 that Ψ_2 is regular over any irrational rotation T.

2) $d = 2, \Psi_3 = (\psi(x), \psi(x + \beta), \psi(x + \gamma))$

As above we reduce the cocycle Ψ_3 to the quotient cocycle $(\Psi_3 + \Delta_3)(x) = (\mathbb{1}_{[0,1-\beta)} - 1 + \beta, \mathbb{1}_{[0,1-\gamma)} - 1 + \gamma)$. Recall that we have seen in subsection 2.3.4 that for α with bounded partial quotients $\Psi_3 + \Delta_3$ is regular and therefore the affine cocycle is also regular when α has bounded partial quotients.

3) $d = 3, \Psi_4 = (\psi(x), \psi(x + \beta), \psi(x + \gamma), \psi(x + \delta))$

Theorem 2.4.3. *There are values of the parameters (β, γ, δ) for which the cocycle is non regular.*

Proof. After reduction by Δ_4, the result follows from Corollary 2.3.17. □

2.4.3 Ergodicity is generic

We consider, as before, the cocycle $\psi(x) = \{x\} - \frac{1}{2}$ and let $S_\beta(x) = x + \beta$ be the rotation by $\beta \in [0, 1)$ on $\mathbb{T}$.

Proposition 2.4.4. *The set $\{(\beta_1, \ldots, \beta_d) \in \mathbb{T}^d : (\psi, \psi \circ S_{\beta_1}, \ldots, \psi \circ S_{\beta_d})$ is ergodic$\}$ is residual.*

Proof. Using Corollary 2.4.2, we only need to show that the set of $(\beta_1,\dots,\beta_d)$ for which the set of accumulation points of $(\{q_n\beta_1\},\dots,\{q_n\beta_d\})_{n\ge 1}$ is dense in $\mathbb{T}^d$, is residual (*i.e.* it includes a dense G_δ subset).

We take $\varepsilon > 0$, $c_1,\dots,c_d \in [0,1)$ and consider the sets $\widetilde{A}_N = \widetilde{A}_N(c_1,\dots,c_d,\varepsilon) := \bigcup_{n=N}^{\infty} A_n(c_1,\dots,c_d,\varepsilon)$, where

$$\begin{aligned} A_n &= A_n(c_1,\dots,c_d,\varepsilon) \\ &:= \{(\beta_1,\dots,\beta_d)\in\mathbb{T}^d:\ \|q_n\beta_1 - c_1\| < \varepsilon,\dots,\|q_n\beta_k - c_d\| < \varepsilon\}. \end{aligned}$$

Clearly $\widetilde{A}_N$ is open and also dense. Fix $0 < \varepsilon_\ell \to 0$. Then the set

$$\bigcap_{\ell\ge 1}\bigcap_{N=1}^{\infty}\widetilde{A}_N(c_1,\dots,c_d,\varepsilon_\ell)$$

is a dense G_δ. Moreover this set equals

$$\{(\beta_1,\dots,\beta_d)\in\mathbb{T}^d : (\exists q_{n_k})\ \ (\{q_{n_k}\beta_1\},\dots,\{q_{n_k}\beta_d\}) \to (c_1,\dots,c_d)\},$$

so the latter set is also a dense G_δ. Therefore the set

$$\bigcap_{(c_1,\dots,c_d)\in\mathbb{Q}^d\cap[0,1)^d}\ \bigcap_{\ell=1}^{\infty}\bigcap_{N=1}^{\infty}\widetilde{A}_N(c_1,\dots,c_k,\varepsilon_\ell)$$

is a dense G_δ and the proof is complete. □

Now, we show that the multiple ergodicity problem has a positive answer for a.a. choices of $(\beta_1,\dots,\beta_d)$. We will need the following classical lemma of Rajchman.

Lemma 2.4.5. *Let $(X,\mathcal{B},\mu)$ be a probability space, $f_n : X \to \mathbb{R}$ such that $f_n \in L^2(X,\mathcal{B},\mu)$, $\|f_n\| < C$, and $f_n \perp f_m$ whenever $n \neq m$. Then $\frac{1}{n}\sum_{k=1}^{n} f_k \to 0$ a.e.*

Proof. It follows from the assumptions that $\sum_{N=1}^{\infty}\|\frac{1}{N^2}\sum_{k=1}^{N^2} f_k\|_2^2 \le \sum_{N=1}^{\infty}\frac{C^2}{N^2} < +\infty$; hence, $\lim_N \frac{1}{N^2}\sum_{k=1}^{N^2} f_k = 0$ a.e.

For $n \ge 1$, let $L_n := [\sqrt{n}]$. We have $L_n^2 \le n < (L_n+1)^2$ and

$$|\frac{1}{n}\sum_{k=1}^{n} f_k| \le \frac{1}{L_n^2}|\sum_{k=1}^{L_n^2} f_k| + 2C\frac{L_n}{n} \underset{n\to\infty}{\longrightarrow} 0, \text{a.e.}$$

□

Proposition 2.4.6. *For every irrational rotation $Tx = x+\alpha$ on $\mathbb{T}$, we have*

$$\mu^{\otimes d}\{(\beta_1,\dots,\beta_d)\in\mathbb{T}^d:\ (\psi,\psi\circ S_{\beta_1},\dots,\psi\circ S_{\beta_d}) \text{ is } T\text{-ergodic}\} = 1.$$

Proof. By Corollary 2.4.2, all we need to show is that the set of $(\beta_1, \dots, \beta_d)$ for which the set of accumulation points of $(\{q_n\beta_1\}, \dots, \{q_n\beta_d\})_{n\geq 1}$ is dense in $\mathbb{T}^d$, is a set of full measure. We will show more: the set of such d-tuples for which $(\{q_n\beta_1\}, \dots, \{q_n\beta_d\})_{n\geq 1}$ is uniformly distributed (mod 1) in $\mathbb{T}^d$ is of full measure.

For almost all $(\beta_1, \dots, \beta_d)$, the sequence $(q_n\beta_1, \dots, q_n\beta_d)_{n\geq 1}$ is uniformly distributed (mod 1). Indeed, by Weyl's criterium of equidistribution (see e.g. [**?**]) it suffices to show that for almost all $(\beta_1, \dots, \beta_d)$ in $\mathbb{T}^d$, for any nontrivial character χ of $\mathbb{T}^d$, the Cesaro averages of the sequence $(\chi(q_n\beta_1, \dots, q_n\beta_d))_{n\geq 1}$ tend to zero.

We have $\chi(q_n\beta_1, \dots, q_n\beta_d) = \exp(2\pi i(s_1 q_n\beta_1 + \dots + s_d q_n \beta_d))$ for integers $s_1, \dots, s_d$. To conclude, we apply Lemma 2.4.5 to $f_n(x_1, \dots, x_d) := \exp(2\pi i(q_n s_1 x_1 + \dots + q_n s_d x_d))$. □

CHAPTER 3

Markov quasi-similarity

3.1 Markov quasi-factors of quasi-discrete spectrum automorphisms

For each $k \in \mathbb{N}$, we denote by $\mathbb{R}_k[\mathbb{X}]$ the space of all real polynomials (of one variable) of degree less than or equal to k.

We will need the following characterization of quasi-eigenfunctions obtained by E. Lesigne in [33].

Theorem 3.1.1. *If $T \in Aut(X, \mathcal{B}, \mu)$ is totally ergodic then the following two conditions are equivalent:*

1. *$f \in (E_k(T))^{\perp}$.*
2. *For μ-a.a. $x \in X$, for each $P \in \mathbb{R}_k[\mathbb{X}]$ and each continuous periodic function Φ on $\mathbb{R}$, we have*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Phi(P(n)) f(T^n x) = 0.$$

Remark 3.1.1. As a matter of fact, the proof of the above theorem from [33] shows that (2) implies (1) without the assumption of total ergodicity of T.

The characterization given in Theorem 3.1.1, will allow us to generalize Theorem 1.9.5 (see Theorem 3.1.4 below).

Lemma 3.1.2. *Assume that* $T \in Aut(X, \mathcal{B}, \mu)$. *Assume moreover that* $\overline{span} \bigcup_{l=0}^{\infty} E_l(T) = L^2(X, \mathcal{B}, \mu)$. *If* $\mathcal{A} \subset \mathcal{B}$ *is a factor such that* $T|_{\mathcal{A}}$ *is totally ergodic then* $T|_{\mathcal{A}}$ *has quasi-discrete spectrum.*

Proof. If $T|_{\mathcal{A}}$ does not have quasi-discrete spectrum then there exists $f \in L^2(\mathcal{A})$ such that $f \in E_k(T|_{\mathcal{A}})^{\perp}$ for all $k > 0$. Since $T|_{\mathcal{A}}$ is totally ergodic, it follows from Theorem 3.1.1 that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Phi(P(n)) f(T^n x) = 0.$$

for all k, $P \in \mathbb{R}_k[\mathbb{X}]$ and Φ. In view of Remark 3.1.1, $f \perp \overline{span} \bigcup_{l=0}^{\infty} E_l(T)$. □

Lemma 3.1.3. *If* T *is totally ergodic,* S *is Markov quasi-factor of it, then* S *is also totally ergodic.*

Proof. The total ergodicity of T is equivalent to the non-existence of non-trivial eigenvalues of U_T of finite order. By Lemma 1.2.4, S cannot have such eigenvalues neither, therefore it is totally ergodic. □

Theorem 3.1.4. *Markov quasi-factor of an automorphism with quasi-discrete spectrum has quasi-discrete spectrum.*

Proof. Let S be a Markov quasi-factor of an automorphism T which has quasi-discrete spectrum. By Theorem 0.1.1 S is a (genuine) factor of $(T \times T \times \ldots, \rho)$ for some $\rho \in J_{\infty}^{e}(T)$. By Remark 1.9.2

$$L^2(X \times X \times \ldots, \rho) = \overline{span} \bigcup_{l \geqslant 0} E_l(T \times T \times \ldots, \rho).$$

We can now apply Lemma 3.1.2 since S is totally ergodic by Lemma 3.1.3. □

3.2 Mixing Markov quasi-similar automorphisms which are not weakly isomorphic

Let us fix $T = T_\sigma$ a standard Gaussian automorphism which is GAG and assume additionally

$$\frac{1}{1-z} \notin L^2(\mathbb{S}^1, \sigma)\ ^{32}; \tag{3.2.1}$$

$$T_\sigma \text{ is a mixing GAG.} \tag{3.2.2}$$

Firstly, we will describe how the two properties of σ can be achieved. We start with T_η an arbitrary mixing GAG (for example, take any simple spectrum mixing Gaussian automorphism, [25]). Then we translate the spectral measure η so that 1 belongs to the topological support of the translation and then symmetrize the measure to obtain a GAG measure σ_1 (see Proposition 11 in [25]) with 1 in the topological support. We have, T_{σ_1} is mixing because σ_1 is still a Rajchman measure. Since 1 is in the support of σ_1, in view of Lemma 5 [24], there is $0 \neq h \in \mathscr{H}_{\sigma_1}$ so that h is not an $L^2(\mathbb{S}^1, \sigma_1)$-coboundary. Finally, take $\sigma = |h|^2\sigma_1 \ll \sigma_1$. Then $\mathbb{1}$ is not an $L^2(\mathbb{S}^1, \sigma)$-coboundary, which, in view of Remark 1.9.3, yields (3.2.1). Since $\sigma \ll \sigma_1$, T_σ is both GAG and mixing.

The process representation of T is denoted by $(P_n)_{n\in\mathbb{Z}}$ and the Gaussian space $H_\sigma = \overline{\text{span}}\{P_n :\ n \in \mathbb{Z}\}$. Set $f = P_0$.

It follows from (3.2.1) that $T_{e^{2\pi i f}}$ is ergodic. In fact it is weakly mixing, so mixing (see Corollary 1.9.7, Remark 1.9.4). As in [24], fix α which is a transcendental complex number of modulus 1 and define the unitary operator $W : L^2(\mathbb{S}^1, \sigma) \to L^2(\mathbb{S}^1, \sigma)$ by setting $(Wj)(z) = g(z)j(z)$, where $g(z) = \alpha$ on the upper half of the circle and $g(z) = \overline{\alpha}$ otherwise. This isometry extends in a unique way to $S \in C^g(T)$.

We will consider now a class of automorphisms which are group extensions of T given by cocycles taking values in $(\mathbb{S}^1)^\mathbb{Z}$:

$$T_{\dots,i_{-1},i_0,i_1,\dots} := T_{\dots,\exp(2\pi i f\circ S^{i_{-1}}),\exp(2\pi i f\circ S^{i_0}),\exp(2\pi i f\circ S^{i_1}),\dots}. \tag{3.2.3}$$

In particular, we will show that automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are Markov quasi-similar but not weakly mixing.

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[32] This is equivalent to saying that $\mathbb{1}$ is not an $L^2(\mathbb{S}^1, \sigma)$-coboundary, or that P_0 is not a Gaussian coboundary.

3.2.1 Coalescence of two-sided cocycle extensions, absence of weak isomorphism

Remark 3.2.1. Assume that $W : (\mathbb{S}^1)^{\mathbb{Z}} \to (\mathbb{S}^1)^{\mathbb{Z}}$ is a continuous homomorphism. Then $W = (\ldots, W_{-1}, W_0, W_1, \ldots)$, where $W_i : (\mathbb{S}^1)^{\mathbb{Z}} \to \mathbb{S}^1$ stands for the composition $proj_i \circ W$ of W with the projection $proj_i$ on the i-th coordinate. Moreover, W_i is a character of $(\mathbb{S}^1)^{\mathbb{Z}}$.

We recall that each character χ of $(\mathbb{S}^1)^{\mathbb{Z}}$ is of the form

$$\chi = \chi^{r_1,\ldots,r_k}_{m_1,\ldots,m_k}, \quad \chi(\ldots z_{-1}, z_0, z_1, z_2, \ldots) = z^{m_1}_{r_1} \cdot \ldots \cdot z^{m_k}_{r_k}$$

for $r_i, m_j \in \mathbb{Z}, \quad 1 \leqslant i, j \leqslant k$.

It follows from [24] that we have the following: [33]

$$\begin{array}{l}\text{the automorphism (3.2.3) is ergodic}\\ \text{(hence weakly mixing and even mixing)}\\ \text{for any sequence of integers } (i_k)_{k\in\mathbb{Z}},\\ \text{provided that } i_k \neq i_l \text{ whenever } k \neq l.\end{array} \tag{3.2.4}$$

Indeed, for no $\chi \neq 1$ we can solve the functional equation

$$\chi(\ldots, \exp(2\pi i f \circ S^{i_{-1}}), \exp(2\pi i f \circ S^{i_0}), \exp(2\pi i f \circ S^{i_1}), \ldots) = \zeta/\zeta \circ T.$$

Equivalently, we cannot solve the functional equation

$$\begin{aligned}&\exp(2\pi i(m_1 f \circ S^{i_{r_1}} + \ldots + m_k f \circ S^{i_{r_k}}))\\ &\quad = \exp(2\pi i f \circ S^{i_{r_1}})^{m_1} \cdot \ldots \cdot \exp(2\pi i f \circ S^{i_{r_k}})^{m_k} = \zeta/\zeta \circ T\end{aligned}$$

which via Proposition 1.9.6 means that $m_1 f \circ S^{i_{r_1}} + \ldots + m_k f \circ S^{i_{r_k}}$ is not a coboundary, equivalently it is not a Gaussian coboundary. Indeed, $m_1 f \circ S^{i_{r_1}} + \ldots + m_k f \circ S^{i_{r_k}}$ is a Gaussian coboundary if and only if [34] $m_1 \alpha^{i_{r_1}} + \ldots + m_k \alpha^{i_{r_k}} = k(z)(1-z)$ for some $k \in L^2(\mathbb{S}^1, \sigma)$ and since the left hand side above is constant different from zero, $\mathbb{1}$ is an $L^2(\mathbb{S}^1, \sigma)$-coboundary ($P_0$ is a Gaussian coboundary) contradicting (3.2.1).

Remark 3.2.2. The following has been proved in [24]: for all $U \in C^g(T)$, $j \in H_\sigma$, $n_1, \ldots, n_t, r \in \mathbb{Z}$ and pairwise distinct integers $p_1, \ldots, p_t$

$$\begin{array}{l}\text{if } n_1 f \circ S^{p_1} + \cdots + n_t f \circ S^{p_t} - f \circ S^r \circ U = j - j \circ T\\ \text{then } t = 1 \text{ and } n_1 = \pm 1.\end{array} \tag{3.2.5}$$

[33] The ergodicity of $T_{\ldots,i_{-1},i_0,i_1,\ldots}$ also follows from the proof of (3.2.5), Proposition 1.9.6, the ergodicity of $T_{e^{2\pi i f}}$ and Remark 1.9.3.

[34] Recall that f in the spectral model corresponds to the constant function $\mathbb{1}$, while S acts by multiplication by g.

We rewrite the above as

$$n_1(g(z))^{p_1} + \cdots + n_t(g(z))^{p_t} - (g(z))^r u(z) = k(z)(1-z),$$

where $u \in \mathscr{H}_\sigma$ is of modulus 1 (and $k \in \mathscr{H}_\sigma$). If we put $Q(z) = n_1 z^{p_1} + \cdots + n_t z^{p_t}$ and $l(z) = Q(g(z)) - (g(z))^r u(z)$ then

$$|l(z)| \geq \Big||Q(g(z))| - 1\Big| = \Big||Q(\alpha)| - 1\Big| \quad \text{for} \quad z \in \mathbb{S}^1.$$

Suppose that $t \geq 2$ or $t = 1$ with $|n_1| \neq 1$. Since α is transcendental, the modulus of $Q(\alpha)$ cannot be equal to 1 [35]. Therefore, there is a constant $A > 0$ such that $|l(z)| > A$. Consequently, the function $\frac{1}{1-z} = k(z)/l(z)$ is in $\mathscr{H}_\sigma$, which contradicts (3.2.1).

Proposition 3.2.1. *Assume that $\bar{i} = (i_k)_{k\in\mathbb{Z}}$ is a strictly increasing sequence of integer numbers. If $(i_k)_{k\in\mathbb{Z}}$ is an arithmetic sequence (progression) then $T_{\bar{i}} = T_{\ldots,i_{-1},i_0,i_1,\ldots}$ is coalescent, that is, each endomorphism commuting with $T_{\bar{i}}$ is invertible.*

Proof. Suppose that $\widetilde{S}$ is an endomorphism commuting with $T_{\bar{i}}$. In view of (1.9.7) and (1.9.8), there exist $U \in C^g(T)$, $\zeta : X_\sigma \to (\mathbb{S}^1)^{\mathbb{Z}}$ measurable and $v : (\mathbb{S}^1)^{\mathbb{Z}} \to (\mathbb{S}^1)^{Z}$ a continuous algebraic epimorphism such that

$$v \circ \psi / \psi \circ U = \zeta / \zeta \circ T, \tag{3.2.6}$$

where

$$\psi = (\ldots, \exp(2\pi i f \circ S^{i_{-1}}), \exp(2\pi i f \circ S^{i_0}), \exp(2\pi i f \circ S^{i_1}), \ldots).$$

The right hand side of (3.2.6) is a function taking values in $(\mathbb{S}^1)^{\mathbb{Z}}$ such that on the r-th coordinate we have

$$\zeta_r / \zeta_r \circ T,$$

where $\zeta = (\zeta_r)_{r\in\mathbb{Z}}$. The left hand side of (3.2.6) is more complicated: it is the multiplication of $v \circ \psi$ with $(\overline{\psi_r} \circ U)$, where $\psi = (\psi_r)_{r\in\mathbb{Z}}$ and $\psi_r = exp(2\pi i f \circ S^{i_r})$. Now (see Remark 3.2.1),

$$X_\sigma \xrightarrow{\psi} (\mathbb{S}^1)^{\mathbb{Z}} \xrightarrow{v} (\mathbb{S}^1)^{\mathbb{Z}} \xrightarrow{proj_r} \mathbb{S}^1.$$

Hence, in view of Remark 3.2.1, each coordinate of the left hand side of (3.2.6) is of the form

$$proj_r \circ v \circ \psi(x) = \psi_{p_1}(x)^{n_1} \cdot \ldots \cdot \psi_{p_t}(x)^{n_t}$$

[35] Indeed, $|Q|^2(\alpha) = 1$, and $|Q|^2$ is an integer coefficients polynomial of degree at least one when $t \geq 2$, so α is algebraic, while when $t = 1$, $|Q|(\alpha) = |n_1| \neq 1$.

(for $r \in \mathbb{Z}$ and the choice of $n_1, \dots, n_t, p_1, \dots, p_t$ depending on r.) Using Proposition 1.9.6, we obtain

$$n_1 f \circ S^{i_{p_1}} + \cdots + n_t f \circ S^{i_{p_t}} - f \circ S^{i_r} \circ U = \ell_r - \ell_r \circ T$$

with some $n_1, \dots, n_t \in \mathbb{Z}$, $\ell_r \in H_\sigma$. By (3.2.5), it follows that $t = 1$ and $n_1 = \pm 1$. Therefore,

$$v\Big((z_r)_{r\in\mathbb{Z}}\Big) = \Big((z_{\pi(r)}^{m_r})_{r\in\mathbb{Z}}\Big), \tag{3.2.7}$$

where $\pi : \mathbb{Z} \to \mathbb{Z}$ and $m_r = \pm 1$ for $r \in \mathbb{Z}$, whence

$$m_r f \circ S^{i_{\pi(r)}} - f \circ S^{i_r} \circ U = \ell_r - \ell_r \circ T.$$

Since $S, U \in C^g(T)$, it follows that

$$m_r f \circ S^{i_{\pi(r)} - i_r} - f \circ U \text{ is a coboundary.}$$

and for $r \neq s$ we obtain that

$$m_r f \circ S^{i_{\pi(r)} - i_r} - m_s f \circ S^{i_{\pi(s)} - i_s} \text{ is also a coboundary.} \tag{3.2.8}$$

However, in view of (3.2.4), $T_{\dots,j_{-1},j_0,j_1,\dots}$ is ergodic for any choice of the sequence (j_k) of distinct integer numbers. Therefore, (3.2.8) implies that

$$i_{\pi(r)} - i_r = const \quad \text{and} \quad m_r = const. \tag{3.2.9}$$

By assumption, there exists $b \in \mathbb{Z} \setminus \{0\}$ such that $i_t = i_0 + tb$ for each $t \in \mathbb{Z}$, whence $i_{t+u} = i_u + tb$. By (3.2.9), $i_{\pi(u)} - i_u = i_{\pi(0)} - i_0$. Therefore,

$$i_{\pi(u)} - i_0 = i_{\pi(u)} - i_u + i_u - i_0 = i_{\pi(0)} - i_0 + i_u - i_0 = \pi(0)b + ub.$$

On the other hand $i_{\pi(u)} - i_0 = \pi(u)b$. Therefore $\pi(u) = \pi(0) + u$ which means that π is a translation on $\mathbb{Z}$. By (3.2.7), v is an automorphism, so finally $\tilde{S}$ is invertible.

□

Similar arguments to those above apply to show the following criterion for the isomorphism of the skew products of the form $T_{\bar{i}}$.

Proposition 3.2.2. *Given two strictly increasing sequences $\bar{i} = (i_k)_{k\in\mathbb{Z}}$ and $\bar{j} = (j_k)_{k\in\mathbb{Z}}$ of integers, the two automorphisms $T_{\bar{i}}$ and $T_{\bar{j}}$ are isomorphic if and only if there exist $m \in \mathbb{Z}$ and a permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ such that $j_{\pi(k)} - i_k = m$ for all $k \in \mathbb{Z}$.*

Absence of weak isomorphism of $T_{\ldots,-1,0,1,2,\ldots}$ and $T_{\ldots,-1,0,2,3,\ldots}$

As an application, consider two extensions $T_{\bar{i}}$, $\bar{i} = (\ldots,-1,0,1,2,\ldots)$ and $T_{\bar{j}}$, $\bar{j} = (\ldots,-1,0,2,3,\ldots)$. They are not isomorphic. Indeed, otherwise there exists $m \in \mathbb{Z}$ and a permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ such that $j_{\pi(k)} = m + i_k = m + k$ for all $k \in \mathbb{Z}$. Therefore, $j_{\pi(-m+1)} = 1$, which is a contradiction.

On the other hand, it has been already noticed in Remark 1.6.1 that whenever an automorphism R is coalescent and R is weakly isomorphic to R' then R is isomorphic to R'. By Proposition 3.2.1, $T_{\ldots,-1,0,1,2,\ldots}$ is coalescent. It follows that $T_{\ldots,-1,0,1,2,\ldots}$ and $T_{\ldots,-1,0,2,3,\ldots}$ are not weakly isomorphic neither.

Remark 3.2.3. Note that not every ergodic automorphism $T_{\ldots,i_{-1},i_0,i_1,\ldots}$ is coalescent. For example, the non-invertible map

$$(x,\underline{z}) \mapsto (S^2x,\ldots,z_{-1},z_0,\overset{0}{z_2},z_3,z_4,\ldots)$$

is an element of the centralizer of $T_{\ldots,-6,-4,-2,0,1,2,3,\ldots}$.

3.2.2 Markov quasi-similarity of two-sided cocycle extensions

Let T be an ergodic automorphism of $(X,\mathcal{B},\mu)$. We take $\varphi : X \to \mathbb{S}^1$ so that the group extension T_φ is ergodic. Then assume that we can find S acting on $(X,\mathcal{B},\mu)$, $S \circ T = T \circ S$ (that is, $S \in C(T)$), such that if we set $G = (\mathbb{S}^1)^{\mathbb{Z}}$ and define

$$\psi : X \to G, \quad \psi(x) = (\ldots,\varphi(S^{-1}x),\overset{0}{\varphi(x)},\varphi(Sx),\varphi(S^2x),\ldots)$$

then T_ψ is ergodic as well. Put now $T_1 = T_\psi$ and let us take a factor T_2 of T_1 obtained by "forgetting" the first $\mathbb{S}^1$-coordinate. In other words, on $(X \times (\mathbb{S}^1)^{\mathbb{Z}}, \mu \otimes m_{(\mathbb{S}^1)^{\mathbb{Z}}})$ we consider two automorphisms

$$T_1(x,\underline{z}) = (Tx,\ldots,z_{-1}\cdot\varphi(S^{-1}x),z_0\cdot\overset{0}{\varphi(x)},z_1\cdot\varphi(Sx),z_2\cdot\varphi(S^2x),\ldots),$$

$$T_2(x,\underline{z}) = (Tx,\ldots,z_{-1}\cdot\varphi(S^{-1}x),z_0\cdot\overset{0}{\varphi(x)},z_1\cdot\varphi(S^2x),z_2\cdot\varphi(S^3x),\ldots),$$

where $\underline{z} = (\ldots,z_{-1},\overset{0}{z_0},z_1,z_2,\ldots)$. For $n \in \mathbb{Z}$ define $I_n : X \times (\mathbb{S}^1)^{\mathbb{Z}} \to X \times (\mathbb{S}^1)^{\mathbb{Z}}$ by setting

$$I_n(x,\underline{z}) = (S^nx,\ldots,z_{n-1},\overset{0}{z_n},z_{n+2},z_{n+3},\ldots).$$

Then I_n is measure-preserving and $I_n \circ T_1 = T_2 \circ I_n$. Therefore

$$U_{T_1} \circ U_{I_n} = U_{I_n} \circ U_{T_2} \tag{3.2.10}$$

with U_{I_n} being an isometry (which is not onto) and

$$U_{I_n}^* F(x, \underline{z}) = \int_{\mathbb{S}^1} F(S^{-n}x, \ldots, \overset{0}{z_{-n}}, \ldots, \overset{n}{z_0}, z, z_1, \ldots)\, dz.$$

Denote by $l_0(\mathbb{Z})$ the subspace of $l^2(\mathbb{Z})$ of complex sequences $\bar{x} = (x_n)_{n\in\mathbb{Z}}$ such that $\{n \in \mathbb{Z} : x_n \neq 0\}$ is finite.

Proposition 3.2.3 ([13], Prop. 3.1)**.** *There exists a nonnegative sequence* $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ *such that* $\sum_{n\in\mathbb{Z}} a_n = 1$ *and*

$$\textit{for every } \bar{x} = (x_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}) \textit{ if } \bar{a} * \bar{x} \in l_0(\mathbb{Z}) \textit{ then } \bar{x} = \bar{0}\ ^{36}. \tag{3.2.11}$$

Let $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be a nonnegative sequence such that $\sum_{n\in\mathbb{Z}} a_n = 1$ and (3.2.11) holds. Let $J : L^2(X \times (\mathbb{S}^1)^{\mathbb{Z}}, \mu \otimes m_{(\mathbb{S}^1)^{\mathbb{Z}}}) \to L^2(X \times (\mathbb{S}^1)^{\mathbb{Z}}, \mu \otimes m_{(\mathbb{S}^1)^{\mathbb{Z}}})$ stand for the Markov operator defined by

$$J = \sum_{n\in\mathbb{Z}} a_n U_{I_n}.$$

In view of (3.2.10), J intertwines U_{T_1} and U_{T_2}.

Denote by $Fin = \mathbb{Z}^{\oplus\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, which is naturally identified with the dual of $(\mathbb{S}^1)^{\mathbb{Z}}$. Let us consider the following two operations on Fin. For $A = (A_s)_{s\in\mathbb{Z}} \in Fin$ (only finitely many $A_s \neq 0$) we set

$$\widehat{A} = (\widehat{A}_s)_{s\in\mathbb{Z}} = \begin{cases} A_s & \text{if } s \leq 0 \\ A_{s-1} & \text{if } s > 1 \\ 0 & \text{if } s = 1 \end{cases}$$

and given $B = (B_s)_{s\in\mathbb{Z}} \in Fin$ such that $B_1 = 0$ we put

$$\widetilde{B} = (\widetilde{B}_s)_{s\in\mathbb{Z}} = \begin{cases} B_s & \text{if } s \leq 0 \\ B_{s+1} & \text{if } s > 0. \end{cases}$$

Of course,

$$\widetilde{\widehat{A}} = A \quad \text{and} \quad \widehat{\widetilde{B}} = B.$$

For $A = (A_s)_{s\in\mathbb{Z}} \in Fin$ and $n \in \mathbb{Z}$ let

$$A + n = ((A+n)_s)_{s\in\mathbb{Z}},$$

where $(A+n)_s = A_{s-n}$ for $s \in \mathbb{Z}$. We have

$$\left(\widehat{A} + n\right)_{n+1} = \widehat{A}_{n+1-n} = \widehat{A}_1 = 0. \tag{3.2.12}$$

[36] $(\bar{a} * \bar{x})_n = \sum_{m=-\infty}^{\infty} a_m x_{n-m}$.

Assume that $B = (B_s)_{s\in\mathbb{Z}} \in Fin$ and $B_{n+1} = 0$; then the element

$$\widetilde{B-n} \text{ is the unique element } C \in Fin \text{ such that } \widehat{C}+n = B. \qquad (3.2.13)$$

Let $\sim$ stand for the equivalence relation in Fin defined by $A \sim B$ if $A = B+n$ for some $n \in \mathbb{Z}$. Denote by Fin_0 a fundamental domain for this relation.

Lemma 3.2.4 (cf. [13])**.** *J has trivial kernel.*

Proof. Each $F \in L^2(X \times (\mathbb{S}^1)^{\mathbb{Z}}, \mu \otimes m_{(\mathbb{S}^1)^{\mathbb{Z}}})$ can be written as

$$F(x,\underline{z}) = \sum_{A\in Fin} f_A(x)A(\underline{z}),$$

where

$$A(\underline{z}) = \Pi_{s\in\mathbb{Z}} z_s^{A_s} \text{ whenever } A = (A_s)_{s\in\mathbb{Z}} \text{ and } f_A \in L^2(X,\mu).$$

Note that $\sum_{A\in Fin} \|f_A\|^2_{L^2(X,\mu)} = \|F\|^2_{L^2(X\times(\mathbb{S}^1)^{\mathbb{Z}},\mu\otimes m_{(\mathbb{S}^1)^{\mathbb{Z}}})}$. Since

$$U_{I_n}(f_A \otimes A)(x,\underline{z}) = (f_A \otimes A)(I_n(x,\underline{z})) = f_A(S^n x)(\widehat{A}+n)(\underline{z}),$$

we have

$$JF(x,\underline{z}) = \sum_{n\in\mathbb{Z}} \sum_{A\in Fin} a_n f_A(S^n x)(\widehat{A}+n)(\underline{z}).$$

By (3.2.12), $(\widehat{A}+n)_{n+1} = 0$, so by changing “the ”index”: substituting $\widehat{A}+n =: B$ and using (3.2.13) (from which it follows that $A = \widetilde{B-n}$), we obtain

$$JF(x,\underline{z}) = \sum_{B\in Fin} \sum_{n\in\mathbb{Z}, B_{n+1}=0} a_n f_{\widetilde{B-n}}(S^n x)B(\underline{z}) = \sum_{B\in Fin} \widetilde{F}_B(x)B(\underline{z}),$$

where $\widetilde{F}_B(x) = \sum_{n\in\mathbb{Z}, B_{n+1}=0} a_n f_{\widetilde{B-n}}(S^n x)$. For every $B \in Fin_0$ and $x \in X$ we define $\xi^B(x) = (\xi^B_n(x))_{n\in\mathbb{Z}}$ by setting

$$\xi^B_{-n}(x) = \begin{cases} f_{\widetilde{B-n}}(S^n x) & \text{if} \quad B_{n+1} = 0 \\ 0 & \text{if} \quad B_{n+1} \neq 0. \end{cases}$$

Therefore, for $k \in \mathbb{Z}$

$$\begin{aligned} \widetilde{F}_{B+k}(x) &= \sum_{n\in\mathbb{Z}, (B+k)_{n+1}=0} a_n f_{\widetilde{B-n+k}}(S^n x) \\ &= \sum_{n\in\mathbb{Z}, B_{(n-k)+1}=0} a_n f_{\widetilde{B-(n-k)}}(S^{-(k-n)}(S^k x)) \\ &= \sum_{n\in\mathbb{Z}} a_n \xi^B_{k-n}(S^k x) = [\bar{a} * \left(\xi^B(S^k x)\right)]_k. \end{aligned}$$

Suppose that $J(F)=0$. It follows that for all $k\in\mathbb{Z}$ and $B\in Fin_0$ we have $[\bar{a}*\left(\xi^B(S^kx)\right)]_k=\widetilde{F}_{B+k}(x)=0$ for μ-a.e. $x\in X$, whence a.s. we also have $[\bar{a}*\left(\xi^B(x)\right)]_k=0$. Letting k run through $\mathbb{Z}$, we obtain that $\bar{a}*\left(\xi^B(x)\right)=\bar{0}$ for μ-a.e. $x\in X$. On the other hand, $\xi^B(x)\in l^2(\mathbb{Z})$ for almost every $x\in X$. In view of (3.2.11), $\xi^B(x)=\bar{0}$ for every $B\in Fin_0$ and for a.e. $x\in X$, hence $f_{\widetilde{A}}=0$ for every $A\in Fin$ with $A_1=0$. It follows that $f_A=0$ for every $A\in Fin$, consequently $F=0$. □

Lemma 3.2.5 (cf. [13])**.** *J^* has trivial kernel.*

Proof. Let

$$F(x,\underline{z})=\sum_{A\in Fin} f_A(x)A(\underline{z}).$$

Then

$$U^*_{I_n}\left(f_A\otimes A\right)(x,\underline{z})=f_A(S^{-n}x)\int_{\mathbb{S}^1} A(\dots,z_{-n},\dots,\overset{n}{z_0},\overset{n+1}{z},\overset{n+2}{z_1},\dots)\,dz.$$

It follows that

$$U^*_{I_n}\left(f_A\otimes A\right)(x,\underline{z})=\begin{cases} f_A(S^{-n}x)\widetilde{A-n}(\underline{z}) & \text{if} \quad A_{n+1}=0\\ 0 & \text{if} \quad A_{n+1}\neq 0.\end{cases}$$

It follows that

$$\begin{aligned} J^*F(x,\underline{z}) &= \sum_{A\in Fin}\ \sum_{n\in\mathbb{Z},A_{n+1}=0} a_n f_A(S^{-n}x)\widetilde{A-n}(\underline{z})\\ &= \sum_{B\in Fin}\sum_{n\in\mathbb{Z}} a_n f_{\widehat{B+n}}(S^{-n}x)B(\underline{z})\\ &= \sum_{A\in Fin,A_1=0}\ \sum_{n\in\mathbb{Z}} a_n f_{A+n}(S^{-n}x)\widetilde{A}(\underline{z}).\end{aligned}$$

Furthermore,

$$\begin{aligned} J^*F(x,\underline{z}) &= \sum_{A\in Fin_0}\ \sum_{k\in\mathbb{Z},(A-k)_1=0}\ \sum_{n\in\mathbb{Z}} a_n f_{A+n-k}(S^{-n}x)\widetilde{A-k}(\underline{z})\\ &= \sum_{A\in Fin_0}\ \sum_{k\in\mathbb{Z},(A-k)_1=0} [\bar{a}*\left(\zeta^A(S^{-k}x)\right)]_k\widetilde{A-k}(\underline{z}),\end{aligned}$$

where $\zeta^A(x)=(\zeta^A_l(x))_{l\in\mathbb{Z}}$ is given by $\zeta^A_l(x)=f_{A-l}(S^lx)$.

Suppose that $J^*(F)=0$. It follows that $[\bar{a}*\zeta^A(S^{-k}x)]_k=0$ for every $A\in Fin_0$ and $k\in\mathbb{Z}$ with $A_{k+1}=0$ and for a.e. $x\in X$. Hence $\bar{a}*\left(\zeta^A(x)\right)\in l_0(\mathbb{Z})$ for μ-a.e. $x\in X$ (the only possibly non-zero terms of the convolved sequence have indices belonging to $\{s\in\mathbb{Z}:\ (A-1)_s\neq 0\}$). Since $\zeta^A(x)\in l^2(\mathbb{Z})$, in view of (3.2.11), $\zeta^A(x)=\bar{0}$ for every $A\in Fin_0$ and for μ-a.e. $x\in X$. Thus $f_A=0$ for all $A\in Fin$ and consequently $F=0$. □

Markov quasi-similarity of $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$

We apply results of this subsection to T a mixing GAG, $\varphi = \exp(2\pi i f)$ and S coming form extension of unitary operator W given by transcendental α. By Lemmas 3.2.4 and 3.2.5, there exists an operator with dense range and trivial kernel intertwining the Koopman operators associated to $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$. It follows that $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are Markov quasi-similar.

Finally, we obtain following theorem

Theorem 3.2.6. *The automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are mixing and Markov quasi-similar but are not weakly isomorphic.*

Proof. By assumption (3.2.2), T is mixing. In view of (3.2.4), both its group extensions $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are ergodic, hence they are also mixing. Moreover, it was shown that they are Markov quasi-similar but not weakly isomorphic. □

Remark 3.2.4. Recalling that a Gaussian mixing automorphism is mixing of all orders, from the result of Rudolph about multiple mixing of isometric extensions (see [44]), it follows that the automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are also mixing of all orders.

Remark 3.2.5. In the beginning of the section, the measure σ was chosen to satisfy (3.2.1) and (3.2.2). Here is another way of specifying it. For a mixing GAG T_η let $\sigma = \eta * \eta$ [37].Then $T = T_\sigma$ is also both mixing and GAG (the latter is unpublished result of F. Parreau). Since the Fourier coefficients of σ are non-negative, $T_{e^{2\pi i P_0}}$ has countable Lebesgue spectrum in the orthocomplement of $L^2(X_\sigma, \mu_\sigma) \otimes \mathbb{1}$ (see Corollary 4 in [24]). Hence P_0 is not a Gaussian coboundary and the conditions (3.2.1) and (3.2.2) hold. Moreover, $\|(P_0)_n\|^2_{L^2(X_\sigma,\mu_\sigma)}$ grows linearly with $|n|$ (recalling that $(P_0)_1 = P_0$, $(P_0)_{n+1} = (P_0)_n + P_0 \circ T^n$ for all $n \in \mathbb{Z}$). Therefore, using the same arguments as in [48, Lemma 4.2], we obtain that the automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ in Theorem 3.2.6 have countable Lebesgue spectrum in the orthocomplement of $L^2(X_\sigma, \mu_\sigma) \otimes \mathbb{1}$.

[37] By $\eta * \eta$ we mean the convolution of the measure η with itself, i.e. $\int_{\mathbb{S}^1} f(z) d\eta * \eta(z) = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} f(v\omega) d\eta(v) d\eta(\omega)$.

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